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Higher gauge fields and fermions in lattice models

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PhD thesis

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Kraków 2022

Wydział Fizyki, Astronomii i Informatyki Stosowanej

Uniwersytet Jagielloński

Oświadczenie

Ja niżej podpisany Błażej Ruba (nr indeksu: 1101308), doktorant Wydziału Fizyki, Astronomii i Informatyki Stosowanej Uniwersytetu Jagiellońskiego oświadczam, że przedłożona przeze mnie rozprawa doktorska pt. „Higher gauge fields and fermions in lattice models" jest oryginalna i przedstawia wyniki badań wykonanych przeze mnie osobiście, pod kierunkiem prof. dr hab. Leszka Hadasza. Pracę napisałem samodzielnie.

Oświadczam, że moja rozprawa doktorska została opracowana zgodnie z Ustawą o prawie autorskim i prawach pokrewnych z dnia 4 lutego 1994 r. (Dziennik Ustaw 1994 nr 24 poz. 83 wraz z późniejszymi zmianami).

Jestem świadom, że niezgodność niniejszego oświadczenia z prawdą ujawniona w dowolnym czasie, niezależnie od skutków prawnych wynikających z ww. ustawy, może spowodować unieważnienie stopnia nabytego na podstawie tej rozprawy.

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Podziękowania

Przede wszystkim chciałbym podziękować mojej żonie Kindze, a także najbliższej rodzinie i przyjaciołom, za to że uczynili moje życie szczęśliwym i za to, że ukształtowali to jaką osobą jestem teraz.

Dziękuję też wszystkim osobom od których miałem okazję się uczyć i z którymi miałem przyjemność współpracować naukowo w czasie moich studiów na Uniwersytecie Jagiellońskim. Wiele z tych osób bardzo przyczyniło się do tego, że te lata były najlepszymi w moim dotychczasowym życiu.

Jestem też wdzięczny mojemu opiekunowi naukowemu Leszkowi Hadaszowi za rady i pomoc w wielu sprawach, naukowych i nie tylko.

Abstract

The purpose of this thesis is to present results concerning higher lattice gauge theory and bosonization, and to summarize the context of this research. Hamiltonian lattice models generalizing both Wilson's lattice gauge theory with finite gauge group and Yetter's Topological Quantum Field Theory are introduced. They are gauge theories in which the gauge group is replaced by an algebraic structure called a crossed module of finite groups. Symmetries and integrable limits of these models are discussed, allowing to formulate general expectations about the dynamics. Much stronger results in this direction are obtained in Euclidean state sum models, both analytically and using Monte Carlo techniques. It is shown that certain factorization takes place, allowing to reduce computation of correlation functions of local observables to more conventional models. More complicated behavior of extended operators sensitive to topology is also studied.

The part about bosonization develops a model which allows to replace any fermionic lattice Hamiltonian, regardless of the spatial dimension, with an equivalent generalized spin system. The latter is subject to constraints, whose understanding takes up a large part of the study. Connections to gauge theory and higher gauge theory are also described.

The thesis consists of four publications, one unpublished manuscript and an introduction.

Streszczenie

Celem tej rozprawy jest prezentacja wyników dotyczących sieciowych wyższych teorii cechowania i bozonizacji, a także podsumowanie kontekstu tych badań. Wprowadzone zostają Hamiltonowskie układy sieciowe będące wspólnym uogólnieniem Wilsonowskich sieciowych teorii cechowania i Topologicznej Kwantowej Teorii Pola Yettera. Są to teorie cechowania w których grupa cechowania jest zastąpiona strukturą algebraiczną zwaną modułem skrzyżowanym grup skończonych. Symetrie i rozwiązywalne układy tych modeli są dyskutowane, co pozwala sformułować ogólne oczekiwania dotyczące dynamiki. Znacznie silniejsze wyniki w tym kierunku są uzyskane w sformułowaniu Euklidesowym, zarówno analitycznie jak i za pomocą metod Monte Carlo. Pokazane jest, że zachodzi pewnego rodzaju faktoryzacja, pozwalająca zredukować obliczanie funkcji korelacji lokalnych obserwabli do bardziej konwencjonalnych modeli. Zbadane jest też bardziej skomplikowane zachowanie rozciągłych operatorów czułych na topologię.

W części dotyczącej bozonizacji rozwijany jest model pozwalający zastąpić dowolny sieciowy Hamiltonian fermionowy, niezależnie od wymiaru przestrzennego, równoważnym uogólnionym układem spinowym. Znaczna część rozważań poświęcona jest badaniu więzów w tych modelach. Zbadane zostały też związki z teoriami cechowania i wyższymi teoriami cechowania.

Rozprawa składa się z czterech publikacji, jednego nieopublikowanego manuskryptu oraz dodatkowego wprowadzenia.

List of publications comprising the thesis

I. A. Bochniak, L. Hadasz and B. Ruba, *Dynamical generalization of Yetter's model based on a crossed module of discrete groups*, Journ. High Energ. Phys. **2021** (2021) 282.

II. A. Bochniak, L. Hadasz, P. Korcyl and B. Ruba, *Dynamics of a lattice 2-group gauge theory model*, Journ. High Energ. Phys. **2021** (2021), 68.

III. A. Bochniak and B. Ruba, *Bosonization based on Clifford algebras and its gauge theoretic interpretation*, Journ. High Energ. Phys. **2020** (2020) 118.

IV. A. Bochniak, B. Ruba, J. Wosiek and A. Wyrzykowski, *Constraints of kinematic bosonization in two and higher dimensions*, Phys. Rev. D **102** (2020) 114502.

The following manuscript is unpublished:

V. A. Bochniak, B. Ruba and J. Wosiek, *Bosonization of Majorana modes and edge states*, arXiv:2107.06335.

In addition, an introduction containing a summary of results and a review of selected background material is included.

Justification of the form of the dissertation

The main part of this thesis is a collection of publications. I would like to explain why this form was chosen. Firstly, my contribution to each of these publications was significant or even decisive, as indicated by the list below. Therefore the included material contains many results that are my own, even though each publication was written jointly with my collaborators. Nevertheless, it does not seem possible to extract only my own part without devoiding it of context. My personal touch to discussed topics may be seen in the way of presentation and the choice of aspects emphasised in the attached introduction.

I. Most new results presented in Publication I were obtained by me.

II. Publication II may be divided into two parts: analytic calculations and numerical studies. I am the principal author of the first part. Moreover, I proposed the general conceptual scheme for simulations, namely a Markov chain with constraint-preserving moves in the set of field configurations. Implementation of this idea and all numerical calculations were performed by another author, dr Piotr Korcyl.

III. My colleague Arkadiusz Bochniak and I have contributed equally to Publication III. My main conceptual contributions include the idea for interpretation of constraint operators as holonomies, and a large portion of algebraic proofs and calculations.

IV. Four authors of Publication IV have contributed equally. My main role in this work was in the algebraic derivation of regularities observed in numerical results, which was done based on the ideas from Publication III.

V. Preprint V is about a generalization of the model studied in Publications III and IV. This idea was proposed by myself. Analytic study of the new bosonization method was carried out jointly with Arkadiusz Bochniak, to which we have contributed equally. The part of preprint V concerning Euclidean representations is due to the other co-author, prof. dr hab. Jacek Wosiek.

Higher gauge fields and fermions in lattice models: introduction to the thesis

Błażej Ruba

Contents

1 Outline

One of the most important questions about many body systems, e.g. lattice models or field theories, is what are their symmetries. It allows to distinguish phases of matter [1], derive conservation laws and constrain possible renormalization group flows, to name a few. Recently it has been emphasized [2, 3] that in many models further results of this nature may be obtained [4] by considering so called higher form symmetries, which act non-trivially only on extended operators. Such symmetries are present in many interesting field theories, e.g. gauge theories and sigma models.

In condensed matter physics, exact higher form symmetries exist only in certain fine-tuned toy models [5, 6], typically Hamiltonian realizations of Topological Quantum Field Theories (TQFTs) [7]. However, it has been argued [8, 9, 10] that robust *emergent* higher form symmetries are among the hallmarks of topologically ordered systems.

Systems with symmetry can be probed theoretically by including background gauge fields. One can also obtain new interesting models by allowing gauge fields to be dynamical, in a procedure often called gauging. In the case of higher symmetries this leads to higher form fields. Interestingly, p-form gauge fields have appeared in the literature much earlier [11] than global higher symmetries.

Ordinary gauge theories are centered around the concept of parallel transport. Similarly, higher gauge fields allow to define "parallel transports" along manifolds of higher dimension, e.g. surfaces. It has been argued [12, 13] that such transports have to be valued in an abelian group. One intuitive way to understand this is that there exists a natural notion of path ordering, but not surface ordering.

There exists a common generalization of 1-form and 2-form gauge theories based on algebraic structures called crossed modules of groups [14] or 2-groups [15]. It involves both 1-form and 2-form degrees of freedom, not independent in general. Gauge fields valued in a crossed module appeared for the first time in Yetter's TQFT [16, 17]. In this construction crossed modules of *finite* groups are used. Yang-Mills-like theories based on crossed modules of Lie groups were proposed in [18, 19], while BF type actions were defined in [20, 21]. More recently, Hamiltonian formulation of Yetter's TQFT was developed in [22, 23, 24].

It has been proposed [25, 26] that topological 2-group gauge theories can be used to describe certain gapped phases of gauge theories. Furthermore, they are supposed to describe Symmetry Protected Topological (SPT) phases with higher symmetries [27], much like the Dijkgraaf-Witten theory [28] is used in the classification of more traditional SPTs [29].

Publication I of the thesis introduces dynamical Hamiltonian lattice models based on crossed modules of finite groups. They generalize both the standard (not topological) 1-form and 2-form gauge theories with finite gauge group. Study of their dynamics is initiated by discussing symmetries and finding four integrable limits. By solving for the space of ground states it shown that they may be described by certain TQFTs, either of Yetter's or Dijkgraaf-Witten class. Besides these results, Publication I contains an extensive introduction to mathematics underlying lattice gauge theories based on crossed modules, including the required algebraic topology and theory of classifying spaces of crossed modules.

Research of dynamics of lattice gauge theories based on crossed modules was continued in Publication II, in which Euclidean state sum formulation was considered. For the purpose of performing Monte Carlo simulations, attention was restricted to four-dimensional periodic cubic lattices and a particular crossed module. Apart from understanding the model at hand, authors wished to better understand what does it mean for a higher form symmetry to be spontaneously broken. Hence suitable order parameters were proposed and studied.

After formulating a constraint-preserving Monte Carlo simulation scheme, numerical calculations were initiated. To the suprise of the authors, they yielded estimates for thermodynamic quantities indicative of a lack of interaction between 1-form and 2-form components of the gauge field. More complicated behavior was seen for order parameters supported on non-contractible lines and surfaces.

Given the clues from numerics, essential factorization into independent 1-form and 2-form gauge models was proven analytically. This result does not depend on the particular choice of crossed module adapted in Publication II. It shows that the structure of a crossed module does not lead to inevitable interactions, in contrast to a famous effect of non-commutativity of the gauge group in Yang-Mills theory. Secondly, combined with generalized Kramers-Wannier dualities [30], it has allowed to obtain a complete phase diagram.

Second part of the thesis (Publications III, IV and Preprint V) is concerned with the subject of bosonization. Bosonization is a set of techniques allowing to replace fermionic degrees of freedom with bosons (or a spin system). There are several motivations to attempt that. Firstly, there exist models which can be solved exactly using these techniques [31, 32]. Secondly, bosonic variables are sometimes easier to implement in numerical investigations, e.g. due to the sign problem. They are also better suited for semiclassical considerations and can lead to nonperturbative insights into strongly coupled systems [33]. Recently bosonization has also been applied in studies of topological phases of matter [34, 35].

The most classical bosonization mapping for lattice models is the Jordan-Wigner transformation [36]. However, this method does not preserve locality of the Hamiltonian in spatial dimensions greater than one.

An alternative was proposed in [37] for cubic lattices in dimension 2 and 3. The idea was to realize commutation relations of fermionic bilinears in a model, here called Gamma model, whose on-site Hilbert spaces are representations of Gamma matrices of sufficiently large dimension. Gamma matrices at distinct lattice sites were assumed to commute. A concrete mapping taking bilinears to products of Gamma matrices was proposed. In order for the considered operators to satisfy certain (anti)commutation rules manifestly obeyed by fermion bilinears, it was necessary to restrict the Hilbert space of the Gamma model by a set of constraints corresponding to closed lattice paths.

In the later work [38] it was established for the first time that solutions of all constraints exist and form a vector space isomorphic to "one half" of the fermionic Fock space, corresponding to one of two possible values of the fermionic parity operator $(-1)^F$. It was remarked that the value of $(-1)^F$ depends on lattice geometry, which was taken to be quite general: arbitrary graph such that every vertex is connected to an even number of neighbors.

The line of research initiated in [37] was continued in Publication III, in which nature of the correspondence between fermions and the Gamma model was elucidated. It was shown that constraint operators may be interpreted as holonomies of a $\mathbb{Z}_2$ gauge field for the fermionic parity symmetry $(-1)^F$. Thus violation of constraints is equivalent to coupling fermions to a background gauge field. Hilbert space of the Gamma model is the direct sum of subspaces corresponding to all possible background $\mathbb{Z}_2$ gauge fields. However, in each subspace only one value of $(-1)^F$ is realized. An explicit relation between $(-1)^F$ and certain function of the $\mathbb{Z}_2$ gauge field was derived.

Solutions of constraints in the Gamma model are highly entangled states, in general difficult to write down in closed form. In the case of two-dimensional toric geometry with $2N \times 2M$ lattice sites, this was achieved in Publication III. States constructed therein are directly related to ground states of Kitaev's toric code [5] and Wen's plaquette model [39].

Another proposal for bosonization in arbitrary dimension was put forward in [40, 41, 42], based on ideas from [34]. It realizes fermions as flux excitations in a higher $\mathbb{Z}_2$ gauge theory with modified Gauss' law. As shown in III, this approach is related to the Gamma model by a simple duality transformation.

Further study of constraints in the Gamma model was undertaken in Publication IV. Exact solutions were found for several small lattice sizes using symbolic algebra software. Furthermore, some bosonized Hamiltonians were diagonalized in the subspace defined by constraints, as well as the subspace corresponding to a constant $\mathbb{Z}_2$ magnetic field.

Further generalization of the Gamma model was introduced in Preprint V. It extends the original approach in three ways.

- Bosonization of Majorana modes is possible.

- Models with multiple fermion flavours (e.g. spins or orbitals) per lattice site are included.
- Lattice sites with odd number of neighbors are allowed.

However, there remains a restriction that the number of Majorana modes[1] on a single lattice site x has to be congruent modulo two to the number of neighbors of x.

First example of a model bosonized with this approach that could not be treated previously is the Hubbard model on a cubic lattice. Curiously, the proposed bosonization mapping treats two spin states in an asymmetric fashion. Nevertheless, the SU(2) rotation symmetry is present on the Gamma model side and acts on-site.

Secondly, one can bosonize Majorana models on lattices with odd coordination number. The simplest example is the honeycomb lattice in two spatial dimensions. It turns out that in this case the proposed approach reduces to the famous solution of Kitaev's honeycomb model [32].

Thirdly, in geometries with boundary there are typically lattice sites whose coordination number is different than in the bulk. Then certain edge modes may be present in the Gamma model. Using the new approach, they are identified and dealt with without difficulties.

The final part of Preprint V is devoted to Monte Carlo study of the Gamma model without constraints. This is motivated by hopes to find new algorithms for simulation of fermions without sign problem.

Section 2 of this document contains a review of background material. Its goal is twofold. Firstly, it explains selected mathematical tools used in the thesis, which may be unfamiliar to many physicists. Secondly, it presents constructions of several models in physics closely related to those studied in the thesis.

Subsection 2.1 deals with rudiments of the theory of principal bundles. The emphasis is laid on global aspects, including the case of dicrete groups and the relation to covering space theory. Afterwards, classifying spaces and characteristic classes are introduced in Subsection 2.2. They are among the most important tools in classifying principal bundles. Their relevance for this thesis stems mostly from their role in Dijkgraaf-Witten and Yetter's theories. The former is discussed in Subsection 2.3. Treatment here is mostly in the spirit of the original article [28], but the discussion of Hilbert spaces and gauge transformations is expanded. To the knowledge of the author, it has not appeared in the literature in this form, although it bears some resemblance to [43]. Afterwards, relations of the Dijkgraaf-Witten theory to some other topics in physics are explained. The main motivation to devote so much space to the Dijkgraaf-Witten theory is that it is largely analogous, but less

[1] Canonical creation-annihilation pair counts as a pair of Majoranas.

technical than Yetter's model. For the sake of clarity, a choice was made to discuss the simpler case in more detail. Subsection 2.4 is devoted to higher symmetries, with U(1) gauge theory serving as the main example. Then Wilson's lattice gauge theories are recalled in Subsection 2.5. This sets the stage to define abelian higher lattice gauge theories and derive a form of Kramers-Wannier duality which was used in Publication IV.

Section 3 is concerned with lattice gauge theories based on crossed modules. First some basic facts about Yetter's TQFT are stated in Subsection 3.1. Instead of presenting a complete construction of the theory, the main new elements (in comparison to the Dijkgraaf-Witten theory) are indicated. A summary of results of Publications I and II is presented in Subsection 3.2. Section 4 is devoted to bosonization. After reviewing the Jordan-Wigner transformation in Subsection 4.1, contents of Publications III, IV and Preprint V are outlined in Subsection 4.2.

2 Background material

2.1 Principal bundles

Let G be a Lie group, not necessarily connected, and M a connected smooth manifold. G-valued gauge field on M is [44] a principal G-bundle $\pi : P \to M$ equipped with a connection. Definitions and selected properties of these objects will now be recalled. Further details can be found in [45, ch. 2].

A fiber bundle over M consists of a manifold E and a smooth map $E \xrightarrow{\pi} M$ such that for some manifold F every $x \in M$ admits a neighbourhood U and a diffeomorphism $U \times F \xrightarrow{\varphi} \pi^{-1}(U)$ satisfying

$$\pi \circ \varphi(y, f) = y \qquad \text{for } y \in U, \ f \in F. \tag{2.1}$$

Sets $\pi^{-1}(x) \subset E$ are submanifolds called fibers of E. Tangent spaces of fibers form a subbundle V of TE, the tangent bundle of E. A connection on E is a subbundle H of TE such that $TE = H \oplus V$.

Principal G-bundle over M consists of a smooth manifold P, a smooth map $P \xrightarrow{\pi} M$ and a smooth right action of G on P such that each $x \in M$ admits a neighbourhood U and a diffeomorphism $U \times G \xrightarrow{\varphi} \pi^{-1}(U)$ such that

$$\pi \circ \varphi(y, g) = y, \qquad \varphi(y, gh) = \varphi(y, g) \cdot h \tag{2.2}$$

for $y \in U$, $g, h \in G$. It is convenient to refer to P itself as a (principal) G-bundle, leaving rest of the structure implicit. Connections on principal bundles are required to be G-invariant. An isomorphism of G-bundles P, P' is a G-equivariant map

$P \xrightarrow{\psi} P'$ such that $P \xrightarrow{\pi} M$ coincides with the composition $P \xrightarrow{\psi} P' \xrightarrow{\pi'} M$. Definition of an isomorphism of G-bundles with connection is self-evident.

In the important special case of discrete G, the above definition states that π is a regular covering with group of deck transformations G [46, ch. 1.3]. In particular universal covers $\widetilde{M} \to M$ are principal G-bundles with $G \cong \pi_1 M$ (fundamental group of M). Another source of examples is provided by homogeneous spaces: if H is a Lie group and $G \subset H$ a closed subgroup, then H is a principal G-bundle over the space H/G of left cosets of G in H.

If S is a manifold on which G acts smoothly from the left, one defines the associated bundle S_P to be the quotient of $P \times S$ by the relation

$$(p \cdot g, s) \sim (p, g \cdot s) \qquad \text{for each } p \in P, g \in G, s \in S. \tag{2.3}$$

If P is equipped with a connection, there is an induced connection on S_P. If S carries additional structure invariant under the G-action, so does S_P. For example,

- S_P is a vector bundle for a linear representation S,
- S_P is a bundle of Lie groups if S is a Lie group on which G acts by automorphisms,
- S_P is a principal S-bundle if S is a Lie group on which G acts as $g \cdot s = \phi(g)s$ for some homomorphism $\phi : G \to S$. To distinguish this example from the previous one, it will be denoted S_P^L.

A special role in the theory is played by the bundle of Lie groups G_P and the bundle of Lie algebras $\mathfrak{g}_P$, where $\mathfrak{g}$ is the Lie algebra of G. They are induced from the adjoint action of G on G and $\mathfrak{g}$, respectively. Significant simplifications occur if G is commutative, for then $G_P = M \times G$ and $\mathfrak{g}_P = M \times \mathfrak{g}$.

Automorphisms of P, sometimes called gauge transformations, may be identified with sections of $G_P \to M$. Thus for G commutative they are the same as smooth functions $M \to G$.

Connections on P form an affine space over the space of $\mathfrak{g}_P$-valued 1-forms. In particular there is a unique connection if G is discrete. To every connection one associates its curvature F, a $\mathfrak{g}_P$-valued 2-form on M.

If $\gamma : [0,1] \to M$ is a (piecewise continuously differentiable) path, connection determines a G-equivariant map $\mathrm{hol}_\gamma : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$, called parallel transport. Let $p_i \in \pi^{-1}(\gamma(i))$, $i = 0, 1$. A general element of $\pi^{-1}(\gamma(i))$ is of the form $p_i \cdot g$ with a unique $g \in G$. Equivariance condition implies that

$$\mathrm{hol}_\gamma(p_0 \cdot g) = p_1 \cdot h_\gamma g \tag{2.4}$$

for some $h_\gamma \in G$ depending on the choice of p_i, but not on g. For a different choice $p_i' = p_i \cdot t_i$ one has

$$\mathrm{hol}_\gamma(p_0' \cdot g) = p_1' \cdot h_\gamma' g \qquad \text{with} \qquad h_\gamma \ = t_1^{-1} h_\gamma t_0. \tag{2.5}$$

Now fix $x \in M$ and choose $p \in \pi^{-1}(x)$. Parallel transport construction assigns a group element h_γ for every loop γ based at x (set $p_0 = p_1 = p$ in the above discussion). Upon a change of p, h_γ is changed only by an overall conjugation

$$h_\gamma \mapsto h_\gamma' = g h_\gamma g^{-1} \tag{2.6}$$

with $g \in G$ independent of γ. Class of the function $\gamma \mapsto h_\gamma$ modulo relation identifying h_γ and h_γ' determines the principal bundle with connection up to isomorphism. See [47, 48] for "inverse" constructions.

The geometrical interpretation of curvature is that it controls variation of parallel transports hol_γ upon variation of γ with fixed endpoints. The extreme case of vanishing curvature is characterized by the property that hol_γ depends only on the homotopy class of γ. Connections with this property are called flat.

For a flat connection, $\gamma \mapsto h_\gamma$ defines a homomorphism $\pi_1 M \to G$. Conversely, every $h \in \mathrm{Hom}(\pi_1 M, G)$ is realized by a principal bundle with connection, say $G^L_{\widetilde{M}}$. Hence isomorphism classes of principal bundles with flat connection are in bijection with $\mathrm{Hom}(\pi_1 M, G)/G$. Such quotients are well studied spaces $\mathcal{M}(M, G)$, called moduli spaces of flat connections. In the notable case of G commutative,

$$\mathcal{M}(M, G) \cong H^1(M, G). \tag{2.7}$$

If M is compact and G is finite (not necessarily abelian), then $\mathcal{M}(M, G)$ is finite.

2.2 Classifying spaces

Consider the problem of classifying principal bundles with connection up to continuous deformations. First note that two connections on a fixed principal G-bundle can be connected by a smooth path.

Secondly, principal bundles are rigid: if $\hat{\pi} : \hat{P} \to [0,1] \times M$ is a principal G-bundle, then its restrictions to $M \times \{t\} \cong M$, are isomorphic.

The problem is reduced to classifying principal G-bundles. One of the most powerful tools for this task is the theory of classifying spaces, which will be sketched below. For more details, consult [49, ch. 4] and [50, ch. 3].

It pays off to depart from the cozy realm of manifolds and smooth maps. Let G be a topological group and M a topological space. Definition of a principal G-bundle

$P \xrightarrow{\pi} M$ is as in Section 2.1, with smoothness conditions replaced by continuity throughout. Concepts of isomorphisms and associated bundles are generalized in a similar fashion.

One shows that smooth principal bundles which are continuously isomorphic are also smoothly isomorphic. Moreover, if G and M are smooth, any continuous principal G-bundle $P \to M$ is isomorphic to a smooth one. Thus differential geometric classification of bundles reduces to (and is generalized by) the topological classification.

Let $P \to M$ be a principal G-bundle and let $f : M' \to M$ be a continuous map. The pullback of P is defined as

$$f^*P = \{(x', p) \in M' \times P \,|\, f(x') = \pi(p)\}. \tag{2.8}$$

It is a principal G-bundle. If M' is paracompact, pullbacks through homotopic maps are isomorphic. Thus one has a map $[f] \mapsto [f^*P]$ from the set $[M', M]$ of homotopy classes of maps $M' \to M$ to the set $\mathrm{Prin}_G(M')$ of isomorphism classes of principal G-bundles on M'.

Principal G-bundle $P \to M$ is called universal if the corresponding map $[M', M] \to \mathrm{Prin}_G(M')$ is a bijection for every CW complex M' (thus in particular for every smooth manifold). It is called n-universal if this condition is satisfied for all CW complexes of dimension $\leq n$.

For a universal principal G-bundle $P \to M$ it is standard to use notation $P = \mathrm{E}G$, $M = \mathrm{B}G$. $\mathrm{B}G$ is called a classifying space of G.

Every topological group admits a universal bundle. Moreover, $\mathrm{B}G$ may be taken to be a CW complex. Then $\mathrm{B}G$ is determined uniquely up to a homotopy equivalence. Indeed, if $\mathrm{E}G \to \mathrm{B}G$ and $\mathrm{E}'G \to \mathrm{B}'G$ are two universal bundles with $\mathrm{B}G$ and $\mathrm{B}'G$ CW complexes, there exist maps $f : \mathrm{B}'G \to \mathrm{B}G$ and $g : \mathrm{B}G \to \mathrm{B}'G$ such that $f^*\mathrm{E}G \cong \mathrm{E}'G$ and $g^*\mathrm{E}'G \cong \mathrm{E}G$, so

$$(f \circ g)^*\mathrm{E}G \cong \mathrm{E}G \cong \mathrm{id}^*_{\mathrm{B}G}\mathrm{E}G \qquad \text{and} \qquad (g \circ f)^*\mathrm{E}'G \cong \mathrm{E}'G \cong \mathrm{id}^*_{\mathrm{B}'G}\mathrm{E}'G, \tag{2.9}$$

which implies that $f \circ g$ and $g \circ f$ are both homotopic to identity.

A principal G-bundle $P \to M$ with M paracompact and $\pi_i P = 0$ for all i (resp. for $i \leq n$) is universal (resp. n-universal). This practical criterion allows to construct many explicit examples.

- Grassmannian of n-planes in $\mathbb{C}^\infty$ is a $\mathrm{BU}(n)$, with $\mathrm{EU}(n) =$ space of n-planes with an orthonormal basis. In particular the infinite dimensional complex projective space $\mathbb{CP}^\infty$ is a $\mathrm{BU}(1)$, with $\mathrm{EU}(1) = \mathbb{S}^\infty$.
  Similarly, there is an m-universal principal $\mathrm{U}(n)$-bundle over the Grassmannian of n-planes in $\mathbb{C}^{n+k}$ for sufficiently large k depending on m.
  There exist analogous constructions for other classical groups.

- If G admits an embedding in some $\mathrm{GL}(n)$ (e.g. G is a compact Lie group), bundles $\mathrm{GL}(n+k)/\mathrm{GL}(k) \to \mathrm{GL}(n+k)/(G \times \mathrm{GL}(k))$ are m-universal for sufficiently large k. Taking a direct limit one may construct a universal bundle.

- Let G be discrete. Eilenberg-MacLane space $K(G,1)$ is defined to be a connected CW complex with fundamental group G and trivial higher homotopy groups. Then $K(G,1)$ is a $\mathrm{B}G$, with its universal covering as $\mathrm{E}G$. Many such $\mathrm{B}G$ are explicitly known. There exists also a general construction [46, Ch. 1.B] of $\mathrm{E}G$ and $\mathrm{B}G$ as a Δ-complexes. The d-simplices of $\mathrm{E}G$ are indexed by d-tuples $(g_0, \dots, g_d) \in G^{d+1}$, and the i-th face of such simplex is $(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_d)$. Group G acts on $\mathrm{E}G$ by simplicial maps, with $g \in G$ taking $(g_0, \dots, g_d)$ to $(gg_0, \dots, gg_d)$. The quotient $\mathrm{E}G/G$ is a $\mathrm{B}G$. Its d-simplices are indexed by tuples $(g_0, \dots, g_d) \in G^{d+1}$ subject to the equivalence relation $(g_0, \dots, g_d) \sim (gg_0, \dots, gg_d)$ for every $g \in G$.

Characteristic class is an assignment of a cohomology class $\alpha(P)$ on a CW complex M to every principal G-bundle $P \to M$. It is required that $\alpha(P)$ depends only on the isomorphism class of P and is natural, in the sense that for a map $f : M' \to M$ one has $\alpha(f^* P) = f^* \alpha(P)$. Then

$$\alpha(P) = f^* \alpha(\mathrm{E}G), \tag{2.10}$$

where $f : M \to \mathrm{B}G$ is a map corresponding to P. Thus characteristic classes determine and are determined by the corresponding elements of cohomology of $\mathrm{B}G$.

The most self-evident application of characteristic classes is to distinguish non-isomorphic bundles: if $\alpha(P) \neq \alpha(P')$, then P, P' are not isomorphic. However, for general G characteristic classes do not provide a complete set of invariants.

Cohomology of $\mathrm{B}G$ is understood for many groups which are frequently encountered in practice, in particular classical Lie groups. Three sources of characteristic classes deserve a special mention.

- For Lie groups G and smooth bundles, Chern-Weil theory [51, ch. XII] constructs a characteristic class $I(F) \in H^\bullet_{\mathrm{dR}}(M)$ from every G-invariant polynomial I on $\mathfrak{g}$. Approximating $\mathrm{B}G$ by smooth manifolds one gets classes in $H^\bullet(\mathrm{B}G, \mathbb{R})$. For compact G this yields an isomorphism of $H^\bullet(\mathrm{B}G, \mathbb{R})$ and the algebra of G-invariant polynomials on $\mathfrak{g}$. For example, $H^\bullet(\mathrm{BU}(n), \mathbb{R})$ is a real polynomial ring in variables $c_1^{\mathbb{R}}, \dots, c_n^{\mathbb{R}}$ called real Chern classes.

- Classifying spaces of some groups can be given cell structure so simple that cohomology can be calculated explicitly [52, ch. 6]. For example, $\mathbb{CP}^\infty$ may be assembled from one cell $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1} \cong \mathbb{C}^n$ in every even dimension $2n$. From this one may deduce that $H^\bullet(\mathrm{BU}(1), \mathbb{Z})$ is an integral polynomial ring in one variable $c_1 \in H^2(\mathrm{BU}(1), \mathbb{Z})$, called the Chern class.

- Let G be discrete and let A be a G-module. Group cohomology [53] $H^\bullet_{\mathrm{grp}}(G, A)$ may be defined as the cohomology with local coefficients [54] $H^\bullet(\mathrm{B}G, \widetilde{A})$, where $\widetilde{A} := A_{\mathrm{E}G}$ is the bundle[2] over $\mathrm{B}G$ associated to A. In the important special case of A being an abelian group with trivial G-action, it reduces to the ordinary (say, singular) cohomology $H^\bullet(\mathrm{B}G, A)$. Alternatively, group cohomology may be defined purely algebraically as $\mathrm{Ext}^\bullet_{\mathbb{Z}G}(\mathbb{Z}, A)$. The two definitions are equivalent, because the simplicial chain complex of $\mathrm{E}G$ is a resolution of $\mathbb{Z}$ free over $\mathbb{Z}G$. This allows to compute group cohomology as the cohomology of an explicit cochain complex

$$\cdots \to C^p(G, A) \xrightarrow{\delta} C^{p+1}(G, A) \to \dots, \tag{2.11}$$

where $C^p(G, A) := \mathrm{Hom}_{\mathbb{Z}G}(C^{\mathrm{simp}}_p(\mathrm{E}G, \mathbb{Z}), A)$ $(= C^p_{\mathrm{simp}}(\mathrm{B}G, A)$ if G acts trivially on $A)$ may be identified with the abelian group of functions $\alpha : G^{p+1} \to A$ satisfying

$$\alpha(gg_0, \dots, gg_p) = g \cdot \alpha(g_0, \dots, g_p). \tag{2.12}$$

The coboundary operator is given by

$$\delta\alpha(g_0, \dots, g_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \alpha(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_{p+1}). \tag{2.13}$$

2.3 Dijkgraaf-Witten theory

2.3.1 Construction

Dijkgraaf-Witten theory [28] is a TQFT whose dynamical field is a G-valued gauge field for some finite group G, or equivalently a map $f : M \to \mathrm{B}G$. In the latter picture homotopies play the role of gauge transformations. The action is given by a characteristic class $\alpha \in H^D(\mathrm{B}G, \mathbb{R}/2\pi\mathbb{Z})$, where D is the spacetime dimension:

$$S = \int_M f^*\alpha. \tag{2.14}$$

Its sign depends on the orientation of M.

It is convenient to choose a base point in M and replace the data of gauge fields by homomorphisms $h \in \mathrm{Hom}(\pi_1 M, G)$. The partition function on a closed manifold M is defined to be

$$Z(M) = \frac{1}{|G|} \sum_{h \in \mathrm{Hom}(\pi_1 M, G)} \exp\left(\mathrm{i} \int_M f_h^* \alpha\right), \tag{2.15}$$

[2] Cohomology with local coefficients in $\widetilde{A}$ may be defined as the sheaf cohomology of the sheaf of sections of $\widetilde{A}$.

where $f_h : M \to \mathrm{B}G$ is determined by h. This sum is finite.

The above definition does not describe phase factors $\mathrm{e}^{\mathrm{i}S}$ very explicitly if $\alpha \neq 0$. Moreover, it is not suitable to deal with amplitudes between states defined on the boundary of M, as required by Atiyah's axioms [7]. For this, it is useful to better understand maps $M \to \mathrm{B}G$.

Pick a triangulation of M and consider the problem of defining a map $M \to \mathrm{B}G$ simplex by simplex, starting from the lowest dimension. Fix a base point $* \in \mathrm{B}G$ and identify $\pi_1(\mathrm{B}G, *)$ with G.

- Since $\mathrm{B}G$ is connected, every map is homotopic to one taking the set $M_0 \subset M$ of vertices (0-simplices of the triangulation) to $*$. Indeed, the homotopy is constructed first on M_0 and then extended to M using the homotopy extension property of $M_0 \subset M$.
- Each oriented edge (1-simplex) e is mapped to a loop in $\mathrm{B}G$ based at $*$, hence represent an element $g_e \in G$. One has $g_{\bar{e}} = g_e^{-1}$, where bar denotes orientation reversal. Such collection $\mathbf{g} = \{g_e\}$ of elements of G indexed by edges will be called a lattice gauge field.
- Consider a triangle (2-simplex) Δ whose three subsequent sides are taken to g_1, g_2, g_3. The boundary of Δ is topologically a circle mapped to a loop of homotopy class $g_3 g_2 g_1$, so the map can be extended to Δ if and only if $g_3 g_2 g_1 = 1$. Lattice gauge field satisfying this condition for every Δ is said to be flat.
- Extensions through higher dimensional simplices exist automatically, since higher homotopy groups of $\mathrm{B}G$ vanish.

Next, one has to understand homotopies. Let $f, f' : M \to \mathrm{B}G$ be given. There is no loss of generality in assuming that both send M_0 to $*$.

- To construct a homotopy between f and f' is to extend to $M \times [0,1]$ the map $M \times \{0,1\} \to \mathrm{B}G$ given by f and f' on $M \times \{0\}$ and $M \times \{1\}$, respectively.
- For every $x \in M_0$ the line segment $\{x\} \times [0,1]$ is sent to $t_x \in \pi_1 \mathrm{B}G$.
- If e is an edge of M with endpoints x, y, oriented from x to y, then extension through $e \times [0,1]$ exists if and only if

  $$g_e' = t_y g_e t_x^{-1}. \tag{2.16}$$

  Thus the collection $\mathbf{t} = \{t_x\}$ of elements of G indexed by M_0 is said to be a gauge transformation from $\mathbf{g}$ to $\mathbf{g}'$, written $\mathbf{t} \cdot \mathbf{g} = \mathbf{g}'$.
- Existence of the extension in subsequent steps is automatic.

The discussion above may be summarized as follows. Every map $M \to \mathrm{B}G$ sending M_0 to $*$ yields a flat lattice gauge field. Conversely, every flat lattice gauge field admits a corresponding map. Two maps are homotopic if and only if the corresponding gauge fields are related by a gauge transformation.

Let $x \in M_0$. Given a gauge field $\mathbf{g}$, the corresponding homomorphism $\pi_1(M,x) \to G$ is constructed as follows. Every loop in M based at x is homotopic to the composition of edges $e_n \cdots e_1$. Define $g_\gamma = g_{e_n} \cdots g_{e_1}$. Flatness of $\mathbf{g}$ guarantees that g_γ depends only on the homotopy class of γ.

One may say more: action of gauge transformations on the set of flat lattice gauge fields has the same orbits as the action of G on $\mathrm{Hom}(\pi_1(M,x),G)$. Stabilizers of this action also agree, because for every $t \in G$ commuting will all g_γ there exists a unique gauge transformation $\mathbf{t}$ with $t_x = t$ such that $\mathbf{t} \cdot \mathbf{g} = \mathbf{g}$. It follows that for every $h \in \mathrm{Hom}(\pi_1(M,x),G)$ there are exactly $|G|^{|X_0|-1}$ flat lattice gauge fields $\mathbf{g}$ corresponding to it.

The question remains how to evaluate (2.14) for a map $f : M \to \mathrm{B}G$ constructed as above. It is convenient to first construct a canonical form for f. As M is built of simplices, this can be achieved by gluing together a coherent family of maps to $\mathrm{B}G$ defined on standard simplices

$$\mathbf{\Delta}^d = \left\{ (x_0, \dots, x_d) \in \mathbb{R}^{d+1} \,\middle|\, x_i \geq 0,\ \sum_{i=0}^{d} x_i = 1 \right\}. \tag{2.17}$$

Let $\mathbf{g}$ be a flat lattice gauge field on $\mathbf{\Delta}^d$. Choose a map $F_{\mathbf{g}} : \mathbf{\Delta}^d \to \mathrm{B}G$ which induces $\mathbf{g}$. Working inductively with respect to d, one may assure that the collection of all $F_{\mathbf{g}}$ satisfies a compatibility condition described below.

Let $\iota : \{0, \dots, k\} \to \{0, \dots, d\}$ be an order-preserving inclusion. Then $\widehat{\iota}$ is defined as the unique convex map $\mathbf{\Delta}^k \to \mathbf{\Delta}^d$ taking j-th basis vector in $\mathbb{R}^{k+1}$ to $\iota(j)$-th basis vector in $\mathbb{R}^{d+1}$. The image of $\widehat{\iota}$ is a face of $\mathbf{\Delta}^d$, so every flat lattice gauge field $\mathbf{g}$ on $\mathbf{\Delta}^d$ restricts to a flat lattice gauge field $\iota^* \mathbf{g}$ on $\mathbf{\Delta}^k$. It is required that $F_{\iota^*\mathbf{g}} = F_{\mathbf{g}} \circ \widehat{\iota}$.

Now let M be a triangulated manifold with a flat lattice gauge field $\mathbf{g}$. Choose an ordering of vertices of M. Then for every D-simplex Ω of M there is a distinguished order-preserving simplicial inclusion $J_\Omega : \mathbf{\Delta}^D \to M$ whose image is Ω. It induces a flat lattice gauge field $J_\Omega^* \mathbf{g}$ on $\mathbf{\Delta}^D$ (by restriction, if one identifies $\mathbf{\Delta}^D$ with its image $\Omega \subset M$). A map $f_\Omega : \Omega \to \mathrm{B}G$ is defined by the condition

$$f_\Omega \circ J_\Omega = F_{J_\Omega^* \mathbf{g}}. \tag{2.18}$$

The compatibility condition satisfied by maps F guarantees that for two D-simplices Ω, Ω' of M one has $f_\Omega|_{\Omega \cap \Omega'} = f_{\Omega'}|_{\Omega \cap \Omega'}$. Therefore there exists a unique $f : M \to \mathrm{B}G$ such that $f_\Omega = f|_\Omega$.

For every Ω let $\epsilon(\Omega) = 1$ or -1, depending on whether orientations of Ω induced by the ordering and orientation of M agree or disagree. Then the fundamental class of M is represented by the chain

$$M = \sum_{\Omega} \epsilon(\Omega)(J_\Omega)_* \mathbf{\Delta}^D, . \tag{2.19}$$

Denoting[3] $\alpha(\mathfrak{g}') := \int_{\mathbf{\Delta}^D} (F_{\mathbf{g}'})^* \alpha$ for a flat lattice gauge field $\mathfrak{g}'$ on $\mathbf{\Delta}^D$, the action may be written in the form

$$S(\mathbf{g}) = \sum_{\Omega} \epsilon(\Omega) \alpha(J_\Omega^* \mathbf{g}). \tag{2.20}$$

Note that even though S depends only on the homotopy class of f and the cohomology class α, the same is not true for the individual terms $\alpha(J_\Omega^* \mathbf{g})$ of the sum (2.20). This is because $\mathbf{\Delta}^D$ is not closed. To make sense of (2.20), a particular cocycle representing the class α has to be chosen. After fixing maps F on standard simplices and the cocycle α, terms of (2.20) are well defined. Each term is a function of the gauge field on a single D-simplex of M. Therefore the whole expression is a manifestly local action, in contrast to the more abstract (2.14).

To finish evaluation of the action, it only remains to understand functions

$$\alpha(\mathfrak{g}) := \int_{\mathbf{\Delta}^D} (F_{\mathbf{g}})^* \alpha \tag{2.21}$$

for $\mathbf{g}$ defined on a standard simplex $\mathbf{\Delta}^D$. If $0 \leq j < i \leq D$, let e_{ij} be the line segment connecting the j-th basis vector of $\mathbb{R}^{D+1}$ to the i-th basis vector. Then e_{ij} are the edges of $\mathbf{\Delta}^D$. Thus a lattice gauge field on $\mathbf{\Delta}^D$ is a collection $\mathbf{g} = \{g_{e_{ij}}\}_{0 \leq j < i \leq D}$. If $\mathbf{g}$ is flat, there exist $h_0, \ldots, h_D \in G$ such that $g_{e_{ij}} = h_i^{-1} h_j$. They are determined by $\mathbf{g}$ uniquely up to $h_i \mapsto h h_i$ (with h independent of i). Hence $\mathbf{g}$ may be equivalently described by a tuple $(h_0, \ldots, h_D)$ modulo equivalence relation

$$(h_0, \ldots, h_D) \sim (hh_0, \ldots, hh_D) \quad \forall h \in G. \tag{2.22}$$

This means that $\alpha(\cdot)$ may be understood as a function $G^{D+1} \to \mathbb{R}/2\pi\mathbb{Z}$ satisfying the homogeneity conditon

$$\alpha(hh_0, \ldots, hh_D) = \alpha(h_0, \ldots, h_D). \tag{2.23}$$

Hence $\alpha \in C^D(G, \mathbb{R}/2\pi\mathbb{Z})$, where G acts trivially on $\mathbb{R}/2\pi\mathbb{Z}$ (recall the discussion around equation (2.12)). By construction, $\delta\alpha = 0$.

Having discussed the action, the partition function on any closed manifold M may be written in the form

$$Z(M) = \frac{1}{|G|^{|M_0|}} \sum_{\mathbf{g}} \prod_{\Omega} \mathrm{e}^{\mathrm{i}\epsilon(\Omega)\alpha(\mathbf{g})}, \tag{2.24}$$

[3]By now α denotes three different, albeit related things: a cohomology class on BG, a cocycle on BG and a function on the set of flat gauge fields on the standard simplex $\mathbf{\Delta}^D$. Hopefully this will not lead to confusion.

where $|M_0|$ is the number of 0-simplices of M. The sum is over all flat lattice gauge fields on M.

The action (2.20) is well defined even if M has a nonempty boundary. In general it is not gauge invariant. However, using the cocycle condition obeyed by α one may show that for every triangulated, oriented smooth $(D-1)$-manifold X with ordered vertices there exists a function I_X such that

$$S(\mathbf{t}\cdot\mathbf{g}) = S(\mathbf{g}) + \sum_i I_{\partial M_i}(\mathbf{t}|_{\partial M_i}, \mathbf{g}|_{\partial M_i}), \tag{2.25}$$

where ∂M_i are connected components of the boundary of M. In general $I_X(\mathbf{t},\mathbf{g})$ depends on a gauge transformation $\mathbf{t}$ and a flat lattice gauge field $\mathbf{g}$ on X. It does not depend on how X is embedded in the boundary of M.

One may explicitly describe I_X in similar terms as for the action[4], showing that I_X is local. This will not be needed in what follows. What is important is that for a pair of gauge transformations $\mathbf{t},\mathbf{t}'$ one may compute $S(\mathbf{t}'\cdot\mathbf{t}\cdot\mathbf{g})$ in two different ways. Since the results have to coincide, one obtains

$$I_X(\mathbf{t}'\mathbf{t},\mathbf{g}) = I_X(\mathbf{t}',\mathbf{t}\cdot\mathbf{g}) + I_X(\mathbf{t},\mathbf{g}). \tag{2.26}$$

Here $\mathbf{t}'\mathbf{t}$ is the self-evident composition of gauge transformations. Such equation is sometimes called Wess-Zumino consistency condition, or simply a cocycle condition.

Gauge-variance of the action may seem disturbing, but since the variation term in (2.25) is supported on the boundary, it may be absorbed by wavefunctions of ingoing and outgoing states[5]. Here is a precise statement. If X is an oriented triangulated $(D-1)$-manifold, let $\mathcal{H}_X^{\mathrm{pre}}$ be the L^2 space on flat lattice gauge fields on X. Modified Gauss' operators on $\mathcal{H}_X^{\mathrm{pre}}$ are defined by

$$(\mathcal{G}_X(\mathbf{t})\psi_X)(\mathbf{g}) = \psi_X(\mathbf{t}^{-1}\cdot\mathbf{g})\exp\left(\mathrm{i}I(\mathbf{t},\mathbf{t}^{-1}\cdot\mathbf{g})\right). \tag{2.27}$$

Clearly they are unitary. Wess-Zumino conditions imply that

$$\mathcal{G}_X(\mathbf{t}')\mathcal{G}_X(\mathbf{t}) = \mathcal{G}_X(\mathbf{t}'\mathbf{t}). \tag{2.28}$$

Let $\mathcal{H}_X$ be the subspace of $\mathcal{H}_X^{\mathrm{pre}}$ of elements invariant to all $\mathcal{G}_X(\mathbf{t})$. This is the physical Hilbert space of Dijkgraaf-Witten theory. It consists of functions satisfying

$$\psi_X(\mathbf{t}\cdot\mathbf{g}) = \psi_X(\mathbf{g})\mathrm{e}^{\mathrm{i}I(\mathbf{t},\mathbf{g})}. \tag{2.29}$$

The orthogonal projection P_X onto $\mathcal{H}_X$ is given by the averaging operation:

$$P_X = \frac{1}{|G|^{X_0}}\sum_{\mathbf{t}}\mathcal{G}_X(\mathbf{t}). \tag{2.30}$$

[4]The only new element is that one has to consider not only maps from standard simplices $\mathbf{\Delta}^d \to \mathrm{B}G$, but also their homotopies $\mathbf{\Delta}^d\times\mathbf{\Delta}^1\to\mathrm{B}G$ corresponding to gauge transformations.

[5]A separate discussion, not presented here, is required for spacetime boundaries which are not time slices, e.g. spatial boundaries.

If $\psi_X \in \mathcal{H}_X$, then $\psi_X(\mathbf{g})$ and $\psi_X(\mathbf{t}\cdot\mathbf{g})$ have the same absolute value. Moreover, if they are nonzero, then the relative phase is uniquely determined by the Gauss' law. It follows that the dimension of $\mathcal{H}_X$ is bounded from above by the number of elements of the moduli space of flat connections $\mathcal{M}(M,G)$. The inequality may be strict for $\alpha \neq 0$, for then it is possible that for some flat lattice gauge field $\mathbf{g}$ one has $I_X(\mathbf{t},\mathbf{g}) \neq 0$ for some $\mathbf{t}$ such that $\mathbf{t}\cdot\mathbf{g} = \mathbf{g}$. In this case the modified Gauss' law (2.29) enforces $\psi_X(\mathbf{g}) = 0$.

Now let M have a boundary decomposed as $\partial M = Y \cup \overline{X}$, bar denoting reversal of the orientation and the ordering of vertices. An operator $\mathcal{O}^{\mathrm{pre}}(M) : \mathcal{H}_X^{\mathrm{pre}} \to \mathcal{H}_Y^{\mathrm{pre}}$ is defined by

$$\langle \psi_Y | \mathcal{O}^{\mathrm{pre}}(M) \phi_X \rangle = \frac{|G|^{\frac{|Y_0|+|X_0|}{2}}}{|G|^{|M_0|}} \sum_{\mathbf{g}} \prod_{\Omega} \mathrm{e}^{\mathrm{i}\epsilon(\Omega)\alpha(J_\Omega^* \mathbf{g})} \overline{\psi_Y(\mathbf{g}|_Y)} \phi_X(\mathbf{g}|_X). \tag{2.31}$$

Here the normalization factor is chosen so that gluing spacetimes along common boundary components corresponds to composition of operators. Furthermore, $\mathcal{O}^{\mathrm{pre}}(\overline{M})$ is the operator adjoint of $\mathcal{O}^{\mathrm{pre}}(M)$.

Gauss' operators have been engineered so that for every gauge transformations $\mathbf{t}_X, \mathbf{t}_Y$ on X and Y one has

$$\mathcal{O}^{\mathrm{pre}}(M) = \mathcal{G}_Y(\mathbf{t}_Y)\mathcal{O}^{\mathrm{pre}}(M) = \mathcal{O}^{\mathrm{pre}}(M)\mathcal{G}_X(\mathbf{t}_X). \tag{2.32}$$

Summing these equalities over all $\mathbf{t}_X$ and $\mathbf{t}_Y$ gives

$$\mathcal{O}^{\mathrm{pre}}(M) = P_Y \mathcal{O}^{\mathrm{pre}}(M) = \mathcal{O}^{\mathrm{pre}}(M) P_X. \tag{2.33}$$

It follows that $\mathcal{O}^{\mathrm{pre}}(M)$ annihilates the orthogonal complement of $\mathcal{H}_X$ in $\mathcal{H}_X^{\mathrm{pre}}$ and has image in $\mathcal{H}_Y$. By restriction, one obtains an operator $\mathcal{O}(M) : \mathcal{H}_X \to \mathcal{H}_Y$.

To complete the construction of the Dijkgraaf-Witten TQFT one has to explain how the dependence on the choice of triangulations (including ordering of vertices) can be lifted. This will be shown now.

From locality and independence of the action on the choice of triangulations in the case of closed M it follows that $\mathcal{O}^{\mathrm{pre}}(M)$, and hence $\mathcal{O}(M)$, are unchanged upon retriangulations of M which do not affect the boundary of M.

Next, for a $(D-1)$-dimensional X let X' be a copy of X, but with a different triangulation. There exists a triangulation of the cylinder $X \times [0,1]$ with boundary $X' \cup \overline{X}$. Denote it by $\mathrm{Cyl}_{X',X}$. By the preceding remarks, $\mathcal{O}(\mathrm{Cyl}_{X',X}) : \mathcal{H}_X \to \mathcal{H}_{X'}$ depends only on X, X', not on the choice of triangulation of the cylinder. Moreover,

$$\mathcal{O}(\mathrm{Cyl}_{X',X})^* \mathcal{O}(\mathrm{Cyl}_{X,X'}) = \mathcal{O}(\mathrm{Cyl}_{X,X}). \tag{2.34}$$

A convenient $\mathrm{Cyl}_{X,X}$ is obtained by subdivision of the product cell structure on $X \times [0,1]$. With such triangulation and with the aid of (2.25) one checks that

$$\mathcal{O}^{\mathrm{pre}}(\mathrm{Cyl}_{X,X}) = \frac{1}{|G|^{|X_0|}} \sum_{\mathbf{t}} \mathcal{G}_X(\mathbf{t}) = P_X, \tag{2.35}$$

so $\mathcal{O}(\mathrm{Cyl}_{X,X}) = 1$. Hence by (2.34), the operator $\mathcal{O}(\mathrm{Cyl}_{X,X'})$ is unitary. It can be used to identify $\mathcal{H}_X$ with $\mathcal{H}_{X'}$ for every X'. With such identifications, $\mathcal{H}_X$ is defined for every oriented, smooth $(D-1)$-dimensional manifold X, without fixing a triangulation. Using the composition law for $\mathcal{O}$ operators one checks that $\mathcal{O}(M) : \mathcal{H}_X \to \mathcal{H}_Y$ remains well-defined with the new meaning of symbols $\mathcal{H}_X, \mathcal{H}_Y$. It does not depend on the choice of triangulation of M.

2.3.2 Relation to other models

Relation of the Dijkgraaf-Witten theory to several interesting models in physics will now be discussed. The list is not exhaustive.

- **Example of functional integration.** Dijkgraaf-Witten theory is a simple quantum field theory model defined by a rigorously constructed path integral. This makes it interesting at least as a toy model. Topological invariance means that the model is a renormalization group fixed point.

- **SPT phases.** Consider a gapped quantum model in spatial dimension $D-1$ with a symmetry G. Even if the system does not exhibit topological order, it might happen that it is not possible to continuously deform the ground state to a product state without either closing the gap or breaking the symmetry. Such phases are called Symmetry Protected Topological or Symmetry Protected Trivial.

  Suppose that G is not anomalous, i.e. it is possible to consistently couple the system to background gauge fields. One is then interested in the response of the system. A natural ansatz is that its long distance properties are described by an effective action for gauge fields which is local and topological.

  Perhaps the most famous example is $D=3$ and $G=\mathrm{U}(1)$, for which topological actions are Chern-Simons functionals parametrized by the level k. Gauss' law in this theory is modified and has a simple interpretation: to each electric charge q there is an associated magnetic flux $\phi = \frac{2\pi}{k} q$. Combining this with the Aharonov-Bohm effect, one may expect excitations with nontrivial exchange phases and spin. In particular transmutation of statistics may take place [55]. Inspection of equations of motion in Chern-Simons theory with external currents shows that k is also related to the Hall conductivity: $\sigma_H = \frac{k}{2\pi}$.

  Dijkgraaf-Witten theory is an analog of Chern-Simons theory for finite groups G. Its actions, parametrized by group cohomology classes, are related to SPT phases with symmetry G. Another argument for such correspondence has been given in [29]. In low dimensions it is now fully established by rigorous index theorems [56]. In general this so-called cohomology classification is expected to be superseded by a classification based on cobordism theory [57].

- **Higher gauge fields.** Dijkgraf-Witten theory is a topological sigma model with target space $K(G,1)$. If G is abelian, there exist also higher Eilenberg-MacLane spaces. These are connected CW complexes $K(G,n)$ with only one nonzero homotopy group, $\pi_n K(G,n) = G$. They have the property that for every CW complex X homotopy classes of maps $X \to K(G,n)$ are in a natural bijection with the cohomology $H^n(X,G)$. Topological sigma models with target space $K(G,n)$ are topological higher (n-form) gauge theories. Such models were studied e.g. in [58].

- **Toric code.** Let X be a triangulated $D-1$ manifold, G a finite group and let $\mathcal{H}$ be the L^2 space on the set of all (not necessarily flat) lattice gauge fields on X. Note that $\mathcal{H}$ is the tensor product of local Hilbert spaces associated to edges of X. Consider the following Hamiltonian on $\mathcal{H}$:

$$H = \sum_{x \in X_0} (1 - A(x)) + \sum_{\Delta} (1 - B(\Delta)). \tag{2.36}$$

  The first sum is over vertices of X. Operator $A(x)$ is defined to be the projection onto states invariant with respect to gauge transformations at x. The second sum is over 2-simplices Δ of X. $B(\Delta)$ is defined to be the projection onto gauge fields flat on Δ.

  All terms of H commute with each other, so the space of ground states is the joint eigenspace of all $A(x)$ and $B(\Delta)$ to eigenvalue 1. Thus it is readily recognized to be the space of states on X of a Dijkgraaf-Witten theory with trivial action $\alpha = 0$. However, the Hilbert space $\mathcal{H}$ contains also excitations, not described by a TQFT.

  The name "toric code" derives from the special case of X being a two-dimensional torus[6]. This model with $G = \mathbb{Z}_2$ was considered first in the field of quantum information [5]. The idea was to use its ground states as qubits. It is not conceivable that gauge invariance and flatness could be imposed as exact constraints in any physical realization, which motivates penalizing their violation energetically instead.

  Many other TQFTs admit realizations similar to the toric code, e.g. via the Levin-Wen construction [6]. Such models are very popular in the field of topological phases.

2.4 Higher symmetries

Consider a Lagrangian field theory with a continuous symmetry G. Noether theorem guarantees the existence of an associated conserved current J, here regarded as a $(D-1)$ form. If $\Sigma \subset M$ is a hypersurface with no boundary, one defines

$$Q(\Sigma) = \int_{\Sigma} J. \tag{2.37}$$

[6]Typically with a standard cubic lattice rather than triangulation, but that does not affect the discussion much.

Notice that $Q(\Sigma)$ may depend on orientation of Σ, sign of the normal vector or both.

Current conservation $\mathrm{d}J = 0$ (up to contant terms) means that $Q(\Sigma)$ is unchanged as Σ is continuously deformed without crossing locations of other operators. Therefore, in a popular jargon, $Q(\Sigma)$ is called a topological operator.

Exponentiation yields operators $U_g(\Sigma)$ labeled by elements $g \in G$. They have the following properties.

- **Ward's identity:** if $\mathbb{S}^{D-1}$ is a sphere around the location x of a local operator $\mathcal{O}^a(x)$, separated from all other insertions, then
$$\langle U_g(\mathbb{S}^{D-1})\mathcal{O}^a(x)\cdots\rangle = \rho(g)^a{}_b\langle\mathcal{O}^b(x)\cdots\rangle, \tag{2.38}$$
where $\langle\cdots\rangle$ denotes quantum averaging and $\rho(g)^a{}_b$ are the matrix elements of the representation in which G acts on local operators.
- **Fusion rule:** if Σ' is a copy of Σ slightly shifted in the normal direction, then
$$U_g(\Sigma')U_h(\Sigma) = U_{gh}(\Sigma). \tag{2.39}$$
- **Neutral element:** $U_1(\Sigma) = 1$.
- **Inversion:** $U_{g^{-1}}(\Sigma) = U_g(\overline{\Sigma})$, where $\overline{\Sigma}$ is Σ with flipped orientations.

Operators $U_g(\Sigma)$ as above can often be constructed also for discrete symmetries, even though Noether theorem is not available in this case. In fact, existence of $U_g(\Sigma)$ is sometimes taken as the definition of a symmetry.

In [3] it was proposed that arbitrary topological operators should be regarded as generalized symmetries. This includes *non-invertible symmetries* described by structures other than groups, e.g. fusion categories [59]. In this work an important role is played by *higher (p-form) symmetries*, corresponding to topological operators associated to submanifolds of codimension $p+1$.

Submanifold of codimension greater than one can be separated arbitrarily far from any given point, without collisions in the intermediate stages. In conjunction with expected locality properties, this means that p-form symmetries with $p > 0$ act trivially on local operators. On the other hand, codimension $p+1$ submanifold may surround an object of dimension p, for example two loops in $\mathbb{R}^3$ can wind nontrivially around each other. Therefore objects charged under a p-form symmetry are of dimension p. Similar considerations suggest that fusion of p-form symmetries has to be commutative for $p > 0$.

An excellent example of higher symmetries is provided by the U(1) gauge theory with the Maxwell action
$$S = \int_M \frac{1}{2g^2} F \wedge *F, \tag{2.40}$$

where F is the curvature form. Shifting the dynamical field by any flat connection (which involves taking the tensor product of corresponding bundles) is a symmetry of the action, called the electric 1-form U(1) symmetry. All local observables are invariant, but holonomies along nontrivial loops in M are changed. It is easy to obtain the conserved current $\frac{1}{g^2} * F$.

If the theory is extended by including dynamical fields with electric charge n, the U(1) symmetry is broken down to $\mathbb{Z}_n$. Electric symmetry exists also in non-abelian gauge theories, in which the symmetry group is the center of the gauge group. For this reason it is typically called the center symmetry [60] in this context. Notably, dynamical consequences of center symmetry have been studied long before the concept of higher symmetries was abstracted.

There exists also so-called magnetic U(1) symmetry, given by operators

$$U^{\mathrm{mag}}_{e^{\mathrm{i}\alpha}}(\Sigma) = \exp\left(\mathrm{i}\alpha \int_\Sigma \frac{1}{2\pi} F\right), \tag{2.41}$$

where Σ is a 2-dimensional oriented surface. If $H^2(M,\mathbb{Z})$ has torsion, slightly more general operators may be constructed: $\exp\left(\mathrm{i}\int_\Sigma c_1\right)$, where c_1 is the Chern class and Σ is a 2-cycle with $\mathbb{R}/2\pi\mathbb{Z}$ coefficients.

Magnetic symmetry is a $(D-3)$-form symmetry. The main charged objects are magnetic monopoles [61] ('t Hooft operators [62]). It has been emphasized [63] that in presence of torsion in cohomology electric and magnetic symmetry operators are not transparent to each other, essentially because there exist flat gauge fields whose Chern class is nontrivial.

For a more general connected gauge group G, magnetic symmetry group is the Pontryagin dual of $\pi_1 G$ (which is the group of magnetic monopole charges [64]). Indeed, in this case $\mathrm{B}G$ is simply-connected, so by the Hurewicz theorem

$$H^2(\mathrm{B}G, \mathbb{R}/2\pi\mathbb{Z}) \cong \mathrm{Hom}(\pi_1 G, \mathbb{R}/2\pi\mathbb{Z}). \tag{2.42}$$

More generally, any characteristic class of degree p gives a $(D-p-1)$-form symmetry.

Electric $J^{\mathrm{el}} = \frac{1}{g^2} * F$ and magnetic $J^{\mathrm{mag}} = \frac{1}{2\pi} F$ currents have quite different roles in the theory. The former is conserved on the account of equations of motion, while conservation of the latter is a purely kinematical statement (Bianchi identity). However, these statements depend on the choice of a particular Lagrangian description of the given field theory. In particular for $D=4$ they are exchanged by the electric-magnetic duality. Moreover, it is not clear if every field theory can be described using a Lagrangian. This is one of motivations to treat all topological operators in field theory on the same footing.

It should be emphasized that existence of the magnetic symmetry relies on continuity properties of fields, which is generally believed to be violated by quantum fields on short scales. In particular, conserved quantities based on topological invariants are often not well defined in discretized field theories. Nevertheless, at least in some models "topological" symmetries are expected to emerge in suitable scaling limits (see e.g. [65]). Remarkably, there exists a recently proposed [66] discretization of the U(1) Maxwell theory that features all symmetries exactly, for all field configurations and at a finite lattice spacing. It was obtained by gauging a subgroup $2\pi\mathbb{Z} \subset \mathbb{R}$ of the electric 1-form symmetry in $\mathbb{R}$ gauge theory, which is rather easy to discretize. In this approach, the Chern class is the holonomy of the (flat) $2\pi\mathbb{Z}$-valued 2-form gauge field. Equally satisfactory[7] formulation of non-abelian lattice gauge theories is not yet available.

2.5 Lattice gauge theory

Lattice gauge theory has been introduced by Wilson [67] as a tool to understand strongly interacting matter, and in that it has achieved tremendous success [68]. This construction will be recalled below.

One starts with a lattice discretizing a Euclidean spacetime M (not necessarily a triangulation) and a compact Lie group G. One G-valued degree of freedom $g_e \in G$ is associated to every oriented lattice edge e. If $\overline{e}$ is e with reversed orientation, then $g_{\overline{e}}$ is taken to be g_e^{-1}.

Elements g_e are subject to gauge transformations as in (2.16). For every lattice face (also called plaquette or 2-cell) f one defines

$$g_{\partial f} = \prod_{e \in \partial f} g_e, \tag{2.43}$$

in which group elements are path-ordered and all edges e are given orientation induced from ∂f. A typical action takes the form

$$S = -\beta \sum_f \operatorname{Re} \operatorname{tr}_R(g_{\partial f}), \tag{2.44}$$

where $\beta > 0$ and R is some faithful representation of G. The action favors configurations with $g_{\partial f}$ close to 1.

Interesting lattice models are obtained even if G is taken to be finite [69, 70]. In contrast to the Dijkgraaf-Witten theory, there is no constraint $g_{\partial f} = 1$. Hence such models do not have an obvious interpretation in terms of continuous fields.

Now let G be commutative. Wilson's construction can be generalized to p-form fields as follows. A degree of freedom $g_\sigma \in G$ is associated to every oriented p-dimensional lattice cell σ, with $g_{\overline{\sigma}} = g_\sigma^{-1}$. A gauge transformation is a collection of

[7] This is a personal opinion of the author.

$t_\gamma \in G$ assigned to oriented $(p-1)$-cells. It acts according to the formula

$$g'_\sigma = g_\sigma \cdot \prod_{\gamma \in \partial\sigma} t_\gamma, \tag{2.45}$$

with γ oriented as induced from σ. Basic invariant observables take the form

$$g_{\partial\omega} = \prod_{\sigma \in \partial\omega} g_\sigma, \tag{2.46}$$

where ω is a $(p+1)$-cell.

General Kramers-Wannier duality [30] relates a p-form lattice gauge theory with finite abelian gauge group G to $(D-p-2)$-form gauge theory with gauge group[8] $G^\vee = \mathrm{Hom}(G, \mathrm{U}(1))$, defined on the dual lattice. This will be derived below.

Consider the partition function

$$Z = \sum_{\{g_\sigma\}} \prod_\omega W(g_{\partial\omega}), \tag{2.47}$$

where W is the exponential of a single term in the action. The product is over all $(p+1)$-cells ω. Fourier representation of W takes the form

$$W(h) = \sum_{\phi \in G^\vee} W^\vee(\phi) \cdot \phi(h). \tag{2.48}$$

Insterting this into the partition function gives

$$Z = \sum_{\{g_\sigma\}} \sum_{\{\phi^\omega\}} \prod_\omega W^\vee(\phi^\omega) \cdot \phi^\omega(g_{\partial\omega}). \tag{2.49}$$

Summation over $\{g_\sigma\}$ may now be performed exactly. Up to an explicit constant depending only on the geometry, it gives a Kronecker delta, which enforces the constraint

$$\prod_{\omega \,:\, \sigma \in \partial\omega} \phi^\omega = 1 \qquad \text{for each } p\text{-cell } \sigma. \tag{2.50}$$

This constraint admits trivial solutions of the form $\phi = \partial\eta$, where

$$(\partial\eta)^\omega = \prod_{\tau \,:\, \omega \in \partial\tau} \eta^\tau. \tag{2.51}$$

All solutions modulo trivial ones form the homology group $H_{p+1}(M, G^\vee)$. If this homology is trivial, one may replace the sum over ϕ by a sum over η, at the cost of an overall constant:

$$Z \propto \sum_{\{\eta^\tau\}} \prod_\omega W^\vee((\partial\eta)^\omega). \tag{2.52}$$

[8]For a finite group G one has $G \cong G^\vee$. However, there is no canonical isomorphism, so it is better to distinguish G from $G^\vee$.

Reinterpreting in terms of the dual lattice[9], the right hand side is the partition function of the dual gauge theory.

More generally, suppose that $H_{p+1}(M, G^\vee)$ is nontrivial. For each homology class h choose a representative $\{\phi_h^\omega\}$. One has

$$Z \propto \sum_{h \in H_{p+1}(M,G^\vee)} \sum_{\{\eta^\tau\}} \prod_\omega W^\vee(\phi_h^\omega(\partial\eta)^\omega). \tag{2.53}$$

From the dual model perspective, h is a flat background gauge field for the electric $(D-p-1)$-form $G^\vee$ symmetry. Summation over background gauge fields in the dual model is often ignored in the literature because it gives a subextensive contribution to thermodynamic quantities.

3 Lattice models based on crossed modules

3.1 Yetter's model

As reviewed in Subsection 2.3, Dijkgraaf-Witten theory may be described as a sigma model with particularly simple target space BG, characterized by only one nonzero homotopy group $\pi_1 = G$, taken to be finite. Yetter's model [16] is a generalization in which the target space $\mathfrak{T}$ has two nonzero homotopy groups, π_1 and π_2. It is assumed that both are finite. The simplest example of such spaces are products of Eilenberg-MacLane spaces

$$\mathfrak{T} = K(\pi_1, 1) \times K(\pi_2, 2), \tag{3.1}$$

corresponding to decoupled 1-form and 2-form topological gauge theories with gauge groups π_1 and π_2, respectively. However, more complicated possibilities exist and have been described in [71]. Product structure (3.1) is replaced by a fibration $\mathfrak{T} \to K(\pi_1, 1)$ with fiber $K(\pi_2, 2)$. Such fibrations are classified by the π_1-module structure on π_2 and the so called Postnikov class

$$\beta \in H^3(K(\pi_1, 1), \widetilde{\pi}_2). \tag{3.2}$$

Up to a weak homotopy equivalence, $\mathfrak{T}$ is classified by the data (π_1, π_2, β), where π_1 is a group, π_2 a π_1-module and β is as above. Description of $\mathfrak{T}$ in these terms was used in the treatment of Yetter's theory in [27].

To describe another perspective on spaces $\mathfrak{T}$ described above, additional algebraic notions will be needed. A crossed module of groups is a quadruple $\mathbb{G} = (\mathcal{E}, \Phi, \triangleright, \Delta)$ consisting of

[9]Strictly speaking, this is possible only if M is orientable or every $g \in G$ satisfies $g^2 = 1$. Otherwise one has to resort to considering orientation-twisted gauge fields, not discussed here. This does not change general conclusions of the analysis.

- groups $\mathcal{E}$, Φ,
- action $\triangleright$ of $\mathcal{E}$ on Φ by automorphisms,
- homomorphism $\Delta : \Phi \to \mathcal{E}$

satisfying Peiffer identities:

$$\begin{aligned} \Delta(\epsilon \triangleright \varphi) &= \epsilon \Delta\varphi \epsilon^{-1}, \\ \Delta\varphi \triangleright \varphi' &= \varphi\varphi'\varphi^{-1}, \end{aligned} \tag{3.3}$$

for $\epsilon \in \mathcal{E}$ and $\varphi, \varphi' \in \Phi$. Important consequences of Peiffer identities include:

- $\ker(\Delta)$ is a central subgroup of Φ.
- $\mathrm{im}(\Delta)$ is a normal subgroup of $\mathcal{E}$ acting trivially on $\ker(\Delta)$. Thus the quotient group $\mathrm{coker}(\Delta) = \Phi/\mathrm{im}(\Delta)$ acts on the abelian group $\ker(\Delta)$.

The notion of a homomorphism $h : \mathbb{G} \to \mathbb{G}' = (\mathcal{E}', \Phi', \triangleright', \Delta')$ is self-evident[10]. Every such h induces group homomorphisms $h_1 : \ker(\Delta) \to \ker(\Delta')$ and $h_2 : \mathrm{coker}(\Delta) \to \mathrm{coker}(\Delta')$. If h_1 and h_2 are isomorphisms, then h is called a weak isomorphism. Weak isomorphisms are not necessarily invertible. Two crossed modules $\mathbb{G}, \mathbb{G}'$ are called weakly equivalent if there exists a zig-zag of weak isomorphisms of the form

$$\mathbb{G} \to \mathbb{G}_1 \leftarrow \dots \to \mathbb{G}_n \leftarrow \mathbb{G}'. \tag{3.4}$$

For every crossed module there exists a corresponding classifying space $\mathrm{B}\mathbb{G}$ (see Appendix C of Publication I for a review of the definition and main properties). Its only nontrivial homotopy groups are $\pi_1 \mathrm{B}\mathbb{G} = \mathrm{coker}(\Delta)$ and $\pi_2 \mathrm{B}\mathbb{G} = \ker(\Delta)$. Every $\mathfrak{T}$ is weakly homotopy equivalent to some $\mathrm{B}\mathbb{G}$. Moreover, (weak) homotopy equivalence $\mathrm{B}\mathbb{G} \to \mathrm{B}\mathbb{G}'$ exists if and only if the crossed modules $\mathbb{G}, \mathbb{G}'$ are weakly equivalent. In particular the data (π_1, π_2, β) describe crossed modules modulo weak equivalence.

As shown by Yetter [16], topological sigma model with target space[11] $\mathrm{B}\mathbb{G}$ may be constructed as a topological gauge theory with fields valued in $\mathbb{G}$, quite similarly as in the Dijkgraaf-Witten theory. In Yetter's work nontrivial actions in $H^D(\mathrm{B}\mathbb{G}, \mathbb{R}/2\pi\mathbb{Z})$ were not considered. This gap was filled by the later work [17].

[10] An important role in the theory of crossed modules is played by a much less obvious notion of weak homomorphisms [72, 73], not used in this work.

[11] In fact Yetter used so called categorical groups rather than crossed modules. It is an easy fact that categorical groups are equivalent to crossed modules, which are preferred here.

3.2 Summary

This section is a brief overview of the first part of the thesis, which consists of two publications about lattice gauge theories based on crossed modules of finite groups:

I A. Bochniak, L. Hadasz and B. Ruba, *Dynamical generalization of Yetter's model based on a crossed module of discrete groups*, Journ. High Energ. Phys. **2021** (2021) 282.

II A. Bochniak, L. Hadasz, P. Korcyl and B. Ruba, *Dynamics of a lattice 2-group gauge theory model*, Journ. High Energ. Phys. **2021** (2021), 68.

In the interest of clarity, some technical details will be omitted in this short summary.

Publication I begins with a self-contained pedagogical introduction to algebraic and topological concepts involved in the subject: groupoids, crossed modules of groupoids and relative homotopy groupoids. Following [22, 24], lattice gauge fields on a lattice X valued in a crossed module $(\mathcal{E}, \Phi, \triangleright, \Delta)$ are defined using this language (Section 2.1 of Publication I). Such gauge field has degrees of freedom $\epsilon_e \in \mathcal{E}$ associated to 1-dimensional cells (edges) and $\varphi_f \in \Phi$ assigned to 2-dimensional cells (plaquettes). They are constrained by the so called *fake flatness* condition

$$\prod_{e \in \partial f} \epsilon_e = \Delta \varphi_f, \tag{3.5}$$

hence only partially independent.

Wilson loops $\epsilon_\gamma \in \mathcal{E}$ are constructed from edge degrees of freedom in the standard way. It is a consequence of fake flatness that their reduction $\overline{\epsilon}_\gamma$ modulo $\mathrm{im}(\Delta)$ is trivial for contractible loops. Thus $\overline{\epsilon}$ is a flat gauge field valued in $\mathrm{coker}(\Delta)$.

Slightly more involved construction of surfaces observables is explained in Section 2.2 of Publication I. It yields an element $\varphi_\sigma \in \ker(\Delta)$ for every spherical surface σ in X. This element depends on degrees of freedom of both types.

In the construction of Yetter's TQFT one considers only field configurations satisfying the flatness constraint $\varphi_\sigma = 1$ for every contractible σ. Then φ gives a homomorphism $\pi_2 X \to \ker(\Delta)$. This flatness condition is not imposed in models studied in Publications I and II.

Afterwards, gauge transformations are discussed. Firstly, there are standard gauge transformations for the ϵ field, parametrized by elements of $\mathcal{E}$ assigned to vertices of X. With respect to these transformations, φ behaves as a matter field.

Secondly, fields of the Yetter's model are subject to transformations parametrized by elements $\psi_e \in \Phi$ assigned to edges. Transformation rule for the φ field has been discusssed in Section 2.3 of Publication I. What will be mentioned here is the transformation law for ϵ:

$$\epsilon_e \mapsto \Delta\psi_e \, \epsilon_e. \tag{3.6}$$

This has the consequence that Wilson loops ϵ_γ are not gauge invariant (not even up to conjugation). Their reductions $\overline{\epsilon}_\gamma$ are invariant, but since they are flat they give only topological observables. In particular standard lattice gauge theory with discrete gauge group (see Section 2.5) can not be recovered from this formalism. For this reason models studied in Publications I and II admit ψ transformations as gauge redundancies only for $\psi_e \in \ker(\Delta)$.

Summarizing, dynamical models generalizing both standard lattice gauge theory and 2-form lattice gauge theory (with finite gauge groups) are obtained from the Yetter's model by introducing two modifications:

- flatness condition on the φ field is lifted, so that nontrivial observables associated to contractible surfaces exist,
- gauge freedom is partially broken, so that nontrivial observables associated to contractible loops exist.

The latter modification has the consequence that models under consideration depend on the whole struture of a crossed module, not only its weak equivalence class.

It is worth to pause for a while to explain the significance of the Postnikov class, as discussed in Appendix C of Publication I. The reduced 1-form gauge field $\overline{\epsilon}$ determines up to homotopy a map $X \xrightarrow{f} K(\mathrm{coker}(\Delta), 1)$. Let $f^*\widetilde{\ker(\Delta)}$ be the pullback through f of the local system $\widetilde{\ker(\Delta)}$ (see Appendix B of Publication I for another formulation more in the spirit of lattice gauge theory). One may also pull back β, yielding a class $f^*\beta \in H^3(X, f^*\widetilde{\ker(\Delta)})$. Furthermore, it is possible to define the curvature $\widehat{\delta}\varphi$ of φ, which is a 3-cocycle with $f^*\widetilde{\ker(\Delta)}$ coefficients. Then $\widehat{\delta}\varphi$ represents the class $f^*\beta$ for any choice of ϵ and φ allowed by the fake flatness constraint for the given $\overline{\epsilon}$. In particular flat φ exists if and only if $f^*\beta$ is trivial.

The main part of Publication I is devoted to the discussion of Hamiltonian models. After specifying the Hilbert space and defining Gauss' operators and various observables, the Hamiltonian is written in the form

$$\mathsf{H} = \mathsf{H}_A + \mathsf{H}_B + \mathsf{H}_V + \mathsf{H}_W, \tag{3.7}$$

in which the first two terms may be regarded as magnetic terms for the 1-form and 2-form, while the latter two are the corresponding elecric terms. This general construction is illustrated by a more explicit discussion for a particular crossed module and a cubic lattice. Afterwards, the symmetries of H are described.

The main result of Publication I, presented in Section 3.6 therein, is a construction of ground state spaces in four integrable limits described by Hamiltonians

$$\mathsf{H}_{\mathrm{E}} = \mathsf{H}_V + \mathsf{H}_W, \quad \mathsf{H}_{AW} = \mathsf{H}_A + \mathsf{H}_W, \quad \mathsf{H}_{\mathrm{M}} = \mathsf{H}_A + \mathsf{H}_B, \quad \mathsf{H}_{BV} = \mathsf{H}_B + \mathsf{H}_V. \tag{3.8}$$

It is shown that in each case the ground state space is the space of states of some TQFT, either of Dijkgraaf-Witten or Yetter type. The effective gauge group (or crossed module of groups) is different in each case. Rather explicit formulas for ground state vectors are also provided.

The dynamics of the full Hamiltonian H interpolates between four (in general distinct) topogical field theories. This picture allows to formulate first conjectures about the phase diagram.

Publication II is concerned with a detailed study of dynamics in the Euclidean state sum formulation. For the most part, periodic four-dimensional cubic lattices and the following crossed module $\mathbb{G}$ are considered:

$$\mathcal{E} = \Phi = \mathbb{Z}_4, \qquad m \triangleright n = (-1)^m n, \qquad \Delta(n) = 2n. \tag{3.9}$$

This crossed module has $\ker(\Delta)$ and $\mathrm{coker}(\Delta)$ both isomorphic to $\mathbb{Z}_2$. Its Postnikov class is the unique nonzero element of $H^3(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

After reviewing observables and gauge transformations, an action is proposed. It has two terms, favoring configurations with ϵ (resp. φ) flat, and two corresponding coupling constants J_1, J_2. The action has several symmetries:

- $\mathbb{Z}_2$ topological charges associated to non-contractible loops,
- an electric 1-form $\mathbb{Z}_2$ symmetry,
- an electric 2-form $\mathbb{Z}_2$ symmetry.

Order parameters for spontaneous breaking of electric symmetries are proposed. For the 1-form symmetry this is a $\mathbb{Z}_4$-valued Polyakov loop (more precisely, the modulus of its average over volume), while for the 2-form symmetry a $\mathbb{Z}_2$-valued "Polyakov surface" observable is used. It is constructed in an ad hoc manner, the discussion of observables associated to non-spherical surfaces being relegated to the Appendix A of Publication II.

The most significant analytic result in Publication II is a factorization theorem, showing that (perhaps up to small finite volume corrections) correlation functions of local gauge-invariant observables may be factorized into correlations in a $\mathbb{Z}_2$ gauge theory with coupling constant J_1 and a 2-form $\mathbb{Z}_2$ gauge theory with coupling constant J_2. The only observables not obeying such factorization are nonlocal order parameters sensitive to topology. This statement is true for any crossed module of finite groups: its proof was presented for the crossed module

(3.9), but it goes through in general case with almost no modification. It shows that the structure of a crossed module is not sufficient to enforce local interaction between 1-form and 2-form fields.

Factorization theorem combined with duality methods allow to reduce description of the phase diagram to understanding of well-studied models: Ising model and Ising gauge theory. Hence the full phase diagram is predicted.

- There is a first order phase transition at $J_1^{\text{crit}} = \frac{1}{2}\text{arsinh}(1) \approx 0.441$. In the phase $J_1 < J_1^{\text{crit}}$ Wilson loops obey the area law and the electric 1-form symmetry is unbroken, while for $J_1 > J_1^{\text{crit}}$ Wilson loops obey a perimeter law and the electric 1-form symmetry is broken.
- There is a second order phase transition at
$$J_2^{\text{crit}} = \frac{1}{2}\text{arsinh}(2\sinh(J_{\text{Ising}}^{\text{crit}})) \approx 0.95, \tag{3.10}$$
where $J_{\text{Ising}}^{\text{crit}} \approx 0.14$ is the critical point of the four-dimensional Ising model. Phases $J_2 < J_2^{\text{crit}}$ and $J_2 > J_2^{\text{crit}}$ are characterized by volume and area laws, respectively, for surface observables. It has also been argued that the electric 2-form symmetry is unbroken for $J_2 < J_2^{\text{crit}}$, but the situation is more complicated for $J_2 > J_2^{\text{crit}}$. One can argue that the symmetry is broken (as in standard 2-form $\mathbb{Z}_2$ gauge theory) if $J_1 > J_1^{\text{crit}}$ and for any J_1 upon restricting to trivial topological charge sector. The argument does not apply for small J_1 and large J_2 in nontrivial topological charge sectors.

In the second part of Publication II, a Monte Carlo scheme suitable for the study of crossed module gauge theory is proposed and applied. This requires the use of constraint-preserving moves in the space of field configurations constructed and studied in Publication I. Besides that, a standard Metripolis algorithm with over-relaxation steps is adapted.

Numerical evaluation of thermodynamic quantities and Polyakov loops gives results in very good agreement with analytic predictions. The most interesting numerical results concern the Polyakov surface, and hence breaking of the 2-form electric symmetry.

- Symmetry is unbroken for $J_2 < J_2^{\text{crit}}$ and broken for $J_1 > J_1^{\text{crit}}$, $J_2 > J_2^{\text{crit}}$.
- In the phase $J_1 < J_1^{\text{crit}}$, $J_2 > J_2^{\text{crit}}$ status of the symmetry depends on the topological charge sector: the symmetry is broken in the trivial topological charge sector, but it becomes restored if nontrivial topological charge is turned on in one of the directions of the Polyakov surface.

It is notable that the most interesting phase $J_1 < J_1^{\text{crit}}$, $J_2 > J_2^{\text{crit}}$ may be interpreted as the basin of attraction of the renormalization group fixed point

$(J_1, J_2) = (0, \infty)$, corresponding to the Yetter's TQFT based on the crossed module $\mathbb{G}$. From this point of view, restoration of the electric symmetry in nontrivial topological charge sectors may be understood by noting that under edge transformations not constrained by the condition $\psi_e \in \ker(\Delta)$, the order parameter is invariant only if topological charges are trivial.

4 Bosonization

4.1 Jordan-Wigner transformation

The subject of bosonization in lattice models goes back to the classic work [36] of Jordan and Wigner. In order to put results of this thesis in context, it is useful to review this construction.

Let us consider a periodic chain with L lattice sites. Each site hosts fermionic operators ϕ_i, ϕ_i^* $(1 \leq i \leq L)$ satisfying canonical anticommutation relations

$$\phi_i \phi_j^* + \phi_j^* \phi_i = \delta_{ij}, \tag{4.1}$$

with $\phi\phi$ and $\phi^*\phi^*$ anticommutators all zero. Boundary conditions are introduced by

$$\phi_{L+1} := (-1)^{\delta} \phi_1, \qquad \phi_{L+1}^* = (-1)^{\delta} \phi_1^* \tag{4.2}$$

with $\delta = 0$ or 1, corresponding to periodic or anti-periodic conditions. Every reasonable Hamiltonian for ϕ fields commutes with the fermionic parity operator

$$(-1)^F = \prod_{i=1}^{L} (-1)^{\phi_i^* \phi_i}. \tag{4.3}$$

Indeed, there is no known process in nature in which the number of fermions changes by an odd number. Thus $(-1)^F$ is an unbreakable $\mathbb{Z}_2$ symmetry.

Pauli operators are defined as

$$\begin{aligned} \sigma_i^x &= \left[\prod_{j<i} (-1)^{\phi_j^* \phi_j} \right] (\phi_i + \phi_i^*), \\ \sigma_i^y &= \left[\prod_{j<i} (-1)^{\phi_j^* \phi_j} \right] \mathrm{i}(\phi_i - \phi_i^*), \\ \sigma_i^z &= (-1)^{\phi_i^* \phi_i}. \end{aligned} \tag{4.4}$$

It is easy to check that they indeed satisfy the algebraic relations of Pauli matrices, with σ_i^a commuting with σ_j^b if $i \neq j$. In this sense σ_i^a are bosonic[12]. It is easy to invert (4.4), which allows to express any Hamiltonian for ϕ in terms of σ.

[12]On the other hand, irreducible representation of the Pauli algebra is finite dimensional. For this reason systems described by Pauli matrices or their generalizations are sometimes called hard core bosons.

Spins σ are subject to the boundary condition

$$\sigma^z_{L+1} = \sigma^z_1, \qquad \sigma^{x,y}_{L+1} = (-1)^{\delta}(-1)^F \sigma^{x,y}_1, \tag{4.5}$$

which depends on $(-1)^F$. This does not lead to significant difficulties, since one may work in subspaces of fixed $(-1)^F$.

Relation (4.4) (as well as its inverse) is nonlocal. However, this nonlocality cancels upon inserting into any local fermionic Hamiltonian H. Indeed, since H is required to commute with $(-1)^F$, it can be expressed in terms of bilinears such as

$$\begin{aligned}
\phi_i^* \phi_i &= \frac{1}{2}(1 - \sigma^z_i), \\
\phi_i \phi_{i+1} &= \frac{1}{4}(\sigma^x_i - \mathrm{i}\sigma^y_i)(\sigma^x_{i+1} - \mathrm{i}\sigma^y_{i+1}), \\
\phi_i^* \phi_{i+1}^* &= \frac{1}{4}(\sigma^x_i + \mathrm{i}\sigma^y_i)(\sigma^x_{i+1} + \mathrm{i}\sigma^y_{i+1}), \\
\phi_i \phi_{i+1}^* &= \frac{1}{4}(\sigma^x_i - \mathrm{i}\sigma^y_i)(\sigma^x_{i+1} + \mathrm{i}\sigma^y_{i+1}), \\
\phi_i^* \phi_{i+1} &= -\frac{1}{4}(\sigma^x_i + \mathrm{i}\sigma^y_i)(\sigma^x_{i+1} - \mathrm{i}\sigma^y_{i+1}).
\end{aligned} \tag{4.6}$$

Jordan-Wigner transformation is not as useful for systems in spatial dimension higher than 1. One may always enumerate fermionic modes $\phi_1, \dots, \phi_L$, but not in a way compatible with the underlying geometry. This has the effect that so called Jordan-Wigner tails $\prod\limits_{j<i}(-1)^{\phi_j^*\phi_j}$ do not cancel in local operators commuting with $(-1)^F$. Thus a different approach is needed.

4.2 Gamma model

Proposal of a locality-preserving bosonization has been put forward in [37]. It was observed therein that the algebra of fermionic operators commuting with $(-1)^F$ can be realized in a generalized spin system with constraints. Mathematical correctness of this approach was later proved in [38]. The idea from [37] was further developed in publications

III A. Bochniak and B. Ruba, *Bosonization based on Clifford algebras and its gauge theoretic interpretation*, Journ. High Energ. Phys. **2020** (2020) 118,

IV A. Bochniak, B. Ruba, J. Wosiek and A. Wyrzykowski, *Constraints of kinematic bosonization in two and higher dimensions*, Phys. Rev. D **102** (2020) 114502,

and the preprint

V A. Bochniak, B. Ruba and J. Wosiek, *Bosonization of Majorana modes and edge states*, arXiv:2107.06335,

which are included as a part of the thesis.

A short summary of constructions in Publication III will now be given. Let $\mathfrak{G} = (V, E)$ be a connected graph such that each vertex has an even number of neighbors. Consider the algebra $\mathcal{A}$ of fermions associated to its vertices. It is generated by pairs $\phi(v), \phi^*(v)$, one for each $v \in V$, satisfying

$$\{\phi(v), \phi(v')\} = \{\phi^*(v), \phi^*(v')\} = 0, \qquad \{\phi(v), \phi^*(v')\} = \delta_{v,v'}. \tag{4.7}$$

Fermionic parity at v is defined by

$$\gamma(v) = 1 - 2\phi^*(v)\phi(v), \tag{4.8}$$

and the total fermionic parity is

$$(-1)^F = \prod_{v \in V} \gamma(v). \tag{4.9}$$

Subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ is defined as the commutant of $(-1)^F$ in $\mathcal{A}$. It is generated by elements $\gamma(v)$ together with hopping operators

$$\mathfrak{s}(e) = (\phi(v) + \phi^*(v))(\phi(v') + \phi^*(v')), \tag{4.10}$$

one for each edge $e \in E$ oriented from v to v'. Relations among generators have been discussed in Section 3 of Publication III. Here only one will be stated:

$$\mathfrak{s}(e_1) \cdots \mathfrak{s}(e_n) = 1 \tag{4.11}$$

for edges $e_1, \ldots, e_n$ forming a loop. It will play a special role soon.

Before proceeding further, consider the effect of coupling fermions (minimally) to a background $\mathbb{Z}_2$ gauge field A. This amounts to replacing each occurence of $\mathfrak{s}(e)$ in the Hamiltonian by

$$\mathfrak{s}_A(e) = (-1)^{A(e)}\mathfrak{s}(e). \tag{4.12}$$

These operators satisfy the same relations as $\mathfrak{s}(e)$, except that (4.11) is replaced by

$$\mathfrak{s}_A(e_1) \cdots \mathfrak{s}_A(e_n) = (-1)^{A(e_1) + \cdots + A(e_n)}. \tag{4.13}$$

The right hand side is a holonomy of the gauge field A, hence depends only on its gauge orbit.

Representation of $\mathcal{A}_0$ in terms of generalized spins is constructed as follows.

- For each $v \in V$ consider the Clifford algebra with generators $\Gamma(v, e)$, one for each edge e incident to v, and one additional generator $\Gamma_*(v)$. By definition, these generators anticommute with each other and square to 1.

- For each v choose an irreducible representation of the corresponding Clifford algebra. This amounts to choosing one of two possible values of $\Gamma_*(v) \prod_e \Gamma(v,e)$.

- Hilbert space $\mathcal{H}$ is defined as the tensor product of Clifford modules associated to vertices. It carries a representation of all local Clifford algebras, with Γ matrices at distinct lattice sites commuting with each other.

- For each edge e choose an orientation and let

$$S(e) = -\mathrm{i}\Gamma(v,e)\Gamma(v',e), \tag{4.14}$$

where $v, v' \in V$ are the two endpoints of e.

- For each gauge orbit $[A]$ of $\mathbb{Z}_2$ gauge fields A let $\mathcal{H}_{[A]} \subset \mathcal{H}$ be the space of elements ψ satisfying

$$S(e_1) \cdots S(e_n)\psi = (-1)^{A(e_1)+\cdots+A(e_n)}\psi \tag{4.15}$$

for every loop $e_1, \ldots, e_n$. Then one has decomposition

$$\mathcal{H} = \bigoplus_{[A]} \mathcal{H}_{[A]} \tag{4.16}$$

and the assignment

$$\gamma(v) \mapsto \Gamma_*(v)|_{\mathcal{H}_{[A]}}, \qquad \mathfrak{s}_A(e) \mapsto S(e)|_{\mathcal{H}_{[A]}} \tag{4.17}$$

defines a representation of $\mathcal{A}_0$ on $\mathcal{H}_{[A]}$.

It was proven in Publication III that $\mathcal{H}_{[A]}$ is an irreducible representation of $\mathcal{H}_{[A]}$, thus isomorphic to a "half" of the Fock space defined by fixing the value of $(-1)^F$. Moreover, the value of $(-1)^F$ is related to A by

$$(-1)^F = (-1)^\alpha \cdot \prod_{e \in E} (-1)^{A(e)}, \tag{4.18}$$

where $\alpha \in \mathbb{Z}_2$ depends on geometry, but not on A. Upon a change of representation of the Clifford algebra on one vertex v, α changes to $\alpha + 1$. It is known how to choose representations to obtain a desired value of α.

Decomposition (4.16) shows that the system with Hilbert space $\mathcal{H}$ describes fermions with all possible background $\mathbb{Z}_2$ gauge fields, with the peculiar relation (4.18) between the number of fermions modulo two and the gauge field. It was shown in Sections 4.3 and 5 of Publication III that $\mathcal{H}$ may be interpreted as the space of gauge invariant states of a $\mathbb{Z}_2$ gauge theory with fermionic matter and a non-standard Gauss' law. Then (4.18) is a consequence of the Gauss' law. Basic operators $\Gamma(v,e)$ create a fermion at v and also certain $\mathbb{Z}_2$ magnetic fluxes in the vicinity of v. Such lump can be a boson due to nontrivial braiding (Aharonov-Bohm phase) of fermions with fluxes.

Now consider the problem of constructing elements of $\mathcal{H}_{[0]}$, corresponding to background gauge field turned off. In fact it is sufficient to find one nonzero element; then the whole space may be built up by acting with operators from $\mathcal{A}_0$. For the geometry of a two-dimensional torus of even length in each direction this has been achieved in Section 4.4 of Publication III by reducing to toric code [5] in two steps:

- fix the value of $\Gamma_*(v)$ for each v to effectively reduce the operator algebra at each lattice site to Pauli algebra,
- perform a unitary rotation whose form alternates from site to site (it is in this step where lattice length being even matters).

Solutions of constraints (4.15) were further studied in Publication IV using symbolic algebra software.

A more general bosonization method has been introduced in the Preprint V. In this construction each vertex v is assumed to host Majorana modes

$$\psi_\alpha(v), \qquad 0 \leq \alpha \leq n(v) \tag{4.19}$$

with some multiplicity $n(v) + 1 > 0$, which has to be congruent modulo 2 to the number of neighbors of v. In this case, the algebra of even fermionic operators is generated by bilinears of two types[13]:

$$\begin{aligned} \mathfrak{s}(e) &= \psi_0(v)\psi_0(v') \qquad \text{for } e \text{ from } v \text{ to } v', \\ \mathfrak{t}_\alpha(v) &= \psi_0(v)\psi_\alpha(v) \qquad \text{for } 1 \leq \alpha \leq n(v) \end{aligned} \tag{4.20}$$

Operators $\mathfrak{s}_A(e)$ are obtained from $\mathfrak{s}(e)$ as in (4.12).

The corresponding spin system is constructed in the following way.

- For each $v \in V$ consider the Clifford algebra with generators $\Gamma(v, e)$, one for each edge e incident to v, and additional generators $\Gamma'_\alpha(v)$ with $1 \leq \alpha \leq n(v)$.
- Again, one has to choose an irreducible representation for each v. Hilbert space $\mathcal{H}$ is their tensor product.
- Operators $S(e)$ and subspaces $\mathcal{H}_{[A]}$ are defined as earlier[14]. Representation of $\mathcal{A}_0$ on $\mathcal{H}_{[A]}$ is defined by

$$\mathfrak{t}_\alpha(v) \mapsto \mathrm{i}\,\Gamma'_\alpha(v)|_{\mathcal{H}_{[A]}}, \qquad \mathfrak{s}_A(e) \mapsto S(e)|_{\mathcal{H}_{[A]}}. \tag{4.21}$$

[13]Unfortunately, there is a clash of notation between Publication III and Preprint V. In this overview notation more similar to that from Publication III is used.

[14]A slightly different convention was adapted in Preprint V: $S(e)$ was defined with opposite sign.

Proofs of validity from Publication III generalize to this setting with no difficulties.

Using the method described above one may treat any fermionic system. Indeed, if the condition that multiplicity of Majoranas matches the number of neighbors at each site is not satisfied, it is possible to bosonize an auxillary system with some spurious fermions. Then they can be removed on the bosonic side by imposing additional constraints. This has been crucial for the treatment of systems with a boundary in Section IV.A of Preprint V.

One interesting fermionic system whose bosonization was made possible by development in Publication V is the Hubbard model. This example was used to demonstrate an important general feature: symmetries acting locally on fermions are still present and act locally after bosonization, despite the fact that Majorana modes ψ_α with $\alpha = 0$ and $\alpha \neq 0$ are treated so differently.

Another interesting connection was found in the consideration of models with one Majorana mode per site on tri-coordinated lattices. It turned out that in this case the proposed bosonization is essentially identical to the one employed in the famous solution of Kitaev's honeycomb model [32].

References

[1] L. Landau, *On The Theory Of Phase Transition*, Zh. Eksp. Teor. Fiz. **7** (1937) 19.

[2] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and duality*, Journ. High Energ. Phys. **2014** (2014) 1.

[3] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized global symmetries*, Journ. High Energ. Phys. **2015** (2015) 172.

[4] D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, *Theta, time reversal and temperature*, Journ. High Energ. Phys. **2017** (2017) 91.

[5] A. Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, Annals Phys. **303** (2003) 2.

[6] X.-G. Wen, *String-net condensation: A physical mechanism for topological phases*, Phys. Rev. B **71** (2005) 045110.

[7] M. F. Atiyah, *Topological quantum field theory*, Publications Mathématiques de I'IHÉS **68** (1988) 175.

[8] M. B. Hastings and X.-G. Wen, *Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance*, Phys. Rev. B **72** (2005) 045141.

[9] Z. Nussinov, G. Ortiz, *A symmetry principle for topological quantum order*, Annals Phys. **324** (2009) 977.

[10] X.-G. Wen, *Emergent anomalous higher symmetries from topological order and from dynamical electromagnetic field in condensed matter systems*, Phys. Rev. B **99** (2019) 205139.

[11] M. Kalb and P. Ramond, *Classical direct interstring action*, Phys. Rev. D **9** (1974) 2273.

[12] C. Teitelboim, *Gauge invariance for extended objects*, Phys. Lett. B **167** (1986) 63.

[13] M. Henneaux and C. Teitelboim, *p-Form electrodynamics*, Found. Phys. **16** (1986) 593.

[14] J. H. C. Whitehead, *Combinatorial homotopy II*, Bull. Amer. Math. Soc. **55** (1949) 453.

[15] J. C. Baez and A. D. Lauda, *Higher dimensional algebra. V: 2-Groups*, *Theory Appl. Categ.* **12**, (2004) 423.

[16] D. N. Yetter, *TQFT's from homotopy* 2*-types*, J. Knot Theor. Ramif. **2** (1993) 113.

[17] J. F. Martins and T. Porter, *On Yetter's invariant and an extension of the Dijkgraaf-Witten invariant to categorical groups*, Theor. Appl. Categories, **18** (2007) 118.

[18] J. C. Baez, *Higher Yang-Mills Theory*, arXiv:hep-th/0206130v2.

[19] H. Pfeiffer, *Higher gauge theory and a non-Abelian generalization of* 2*-form electrodynamics*, Annals Phys. **308** (2003) 447.

[20] F. Girelli, H. Pfeiffer and E. Popescu, *Topological higher gauge theory: From BF to BFCG theory*, J. Math. Phys. **49** (2008) 032503.

[21] J. Faria Martins and A. Miković, *Lie crossed modules and gauge-invariant actions for 2-BF theories*, Adv. Theor. Math. Phys. **15** (2011) 1059.

[22] A. Bullivant, M. Calçada, Z. Kádár, P. Martin and J. F. Martins, *Topological phases from higher gauge symmetry in* $3+1$ *dimensions*, Phys. Rev. B **95** (2017) 155118.

[23] C. Delcamp and A. Tiwari, *From gauge to higher gauge models of topological phases*, Journ. High Energ. Phys. **2018** (2018) 49.

[24] A. Bullivant, M. Calçada, Z. Kádár, J. F. Martins and P. Martin, *Higher lattices, discrete two-dimensional holonomy and topological phases in* $(3+1)D$ *with higher gauge symmetry*, Rev. Math. Phys. **32** (2020) 2050011.

[25] S. Gukov and A. Kapustin, *Topological Quantum Field Theory, Nonlocal Operators, and Gapped Phases of Gauge Theories*, arXiv:1307.4793[hep-th].

[26] A. Kapustin and R. Thorngren, *Topological field theory on a lattice,discrete theta-angles and confinement*, Adv. Theor. Math. Phys. **18** (2014) 1233.

[27] A. Kapustin and R. Thorngren, *Higher Symmetry and Gapped Phases of Gauge Theories*, *Prog. Math.* **324** (2017) 177–202.

[28] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, *Commun. Math. Phys.* **129** (1990) 393.

[29] X. C. Chen et al., *Symmetry protected topological orders and the group cohomology of their symmetry group*, Phys. Rev. B **87** (2013) 155114.

[30] H. A. Kramers and G. H. Wannier, *Statistics of the Two-Dimensional Ferromagnet. Part I*, Phys. Rev. **60** (1941) 252.

[31] T. D. Schultz, D. C. Mattis and E. H. Lieb, *Two-Dimensional Ising Model as a Soluble Problem of Many Fermions*, Rev. Mod. Phys. **36** (1964) 856.

[32] A. Kitaev, *Anyons in an exactly solved model and beyond*, Annals Phys. **321** (2006) 2.

[33] A. O. Gogolin, A. A. Nersesyan and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems*, Cambridge University Press 1999.

[34] D. Gaiotto and A. Kapustin, *Spin TQFTs and fermionic phases of matter*, Int. J. Mod. Phys. A **31** (2016) 1645044.

[35] A. Kapustin and R. Thorngren, *Fermionic SPT phases in higher dimensions and bosonization*, Journ. High Energ. Phys. **2007** (2007) 80.

[36] P. Jordan and E. Wigner *Über das Paulische Äquivalenzverbot*, Z. Phys. **47** (1928) 631.

[37] J. Wosiek *A local representation for fermions on a lattice*, Acta Phys. Pol. B **13** (1982) 543.

[38] A. M. Szczerba, *Spins and fermions on arbitrary lattices*, Commun. Math. Phys. **98** (1985) 513.

[39] X.-G. Wen, *Quantum Orders in an Exact Soluble Model*, Phys. Rev. Lett. **90** (2003) 016803.

[40] Y-A. Chen, A. Kapustin and Đ. Radičević, *Exact bosonization in two spatial dimensions and a new class of lattice gauge theories*, Ann. Phys. **393** (2018) 234.

[41] Y-A. Chen and A. Kapustin, *Bosonization in three spatial dimensions and a 2-form gauge theory* Phys. Rev. B **100** (2019) 245127.

[42] Y.-A. Chen, *Exact bosonization in arbitrary dimensions*, Phys. Rev. Research **2** (2020) 033527.

[43] D. N. Yetter, *Topological quantum field theories associated to finite groups and crossed G-sets*, J. Knot Theory Ramif. **01** (1992), 1.

[44] T.-T. Wu and C.-N. Yang, *Concept of nonintegrable phase factors and global formulation of gauge fields*, Phys. Rev. D **12** (1975) 3845.

[45] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Vol. 1*, Interscience Publishers 1963.

[46] A. Hatcher, *Algebraic Topology*, Cambridge University Press 2002.

[47] J. W. Barrett, *Holonomy and path structures in general relativity and Yang-Mills theory*, Int. J. Theor. Phys. **30** (1991), 1171.

[48] U. Schreiber and K. Waldorf, *Parallel Transport and Functors*, J. Homotopy Relat. Struct. **4** (2009), 187.

[49] D. Husemoller, *Fibre Bundles*, Springer 1994.

[50] G. Rudolph and M. Schmidt, *Differential Geometry and Mathematical Physics, Part II. Fibre Bundles, Topology and Gauge Fields*, Springer 2017.

[51] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Vol. 2*, Interscience Publishers, 1969.

[52] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press 1974.

[53] K. Brown, *Cohomology of groups*, Springer (1982).

[54] N. E. Steenrod, *Homology With Local Coefficients*, Annals Math. **44** (1943), 610.

[55] F. Wilczek, *Quantum Mechanics of Fractional-Spin Particles*, Phys. Rev. Lett. **49** (1982), 957.

[56] Y. Ogata, *A $H^3(G,\mathbb{T})$-valued index of symmetry protected topological phases with on-site finite group symmetry for two-dimensional quantum spin systems*, arXiv:2101.00426.

[57] A. Kapustin, *Symmetry Protected Topological Phases, Anomalies, and Cobordisms: Beyond Group Cohomology*, arXiv:1403.1467.

[58] S. Monnier, *Higher Abelian Dijkgraaf-Witten Theory*, Lett. Math. Phys. **105** (2015) 1321.

[59] L. Bhardwaj, Y. Tachikawa, *On finite symmetries and their gauging in two dimensions*, Journ. High Energ. Phys. **2018** (2018) 189.

[60] K. Holland and U.-J. Wiese, *The Center symmetry and its spontaneous breakdown at high temperatures* in M. Shifman, *At The Frontier of Particle Physics. Handbook of QCD. Vol. 3* (2001) 1909.

[61] P. A. M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. Roy. Soc. Lond. A **133** (1931) 60.

[62] G. 't Hooft, *On the phase transition towards permanent quark confinement*, Nucl. Phys. B **138** (1978) 1.

[63] D. S. Freed, G. W. Moore and G. Segal, *Heisenberg groups and noncommutative fluxes*, Annals Phys. **322** (2007) 236.

[64] P. Goddard, J. Nuyts and D. Olive, *Gauge theories and magnetic charge*, Nucl. Phys. B **125** (1977), 1.

[65] M. Luschër, *Topology of lattice gauge fields*, Commun. Math. Phys. **85** (1982) 39.

[66] T. Sulejmanpasic and C. Gattringer, *Abelian gauge theories on the lattice: θ-Terms and compact gauge theory with(out) monopoles*, Nucl. Phys. B **943** (2019) 114616.

[67] K. G. Wilson, *Confinement of quarks*, *Phys. Rev. D* **10** (1974) 2445.

[68] S. Hashimoto and S. R. Sharpe, *Lattice Quantum Chromodynamics*, in: P. A. Zyla et al., *Review of Particle Physics*, Prog. Theor. Exp. Phys. **2020** (2020), 083C01.

[69] F. J. Wegner, *Duality in Generalized Ising Models and Phase Transitions Without Local Order Parameters*, *J. Math. Phys.* **12** (1971) 2259.

[70] M. Creutz and L. Jacobs and C. Rebbi *Experiments with a gauge invariant Ising system*, Phys. Rev. Lett. **42** (1979) 1390.

[71] S. MacLane and J. .H. C. Whitehead, *On the* 3*-type of a complex*, Proc. Nat. Acad. Sci. USA **36** (1950) 41.

[72] B. Noohi, *Notes on 2-groupoids, 2-groups and crossed modules*, Homol. Homotopy Appl. **9** (2007), 75.

[73] B. Noohi, *On weak maps between 2-groups*, arXiv:math/0506313.

Published for SISSA by Springer

Received: *November 2, 2020*
Accepted: *February 19, 2021*
Published: *March 30, 2021*

Dynamical generalization of Yetter's model based on a crossed module of discrete groups

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Abstract: We construct a lattice model based on a crossed module of possibly non-abelian finite groups. It generalizes known topological quantum field theories, but in contrast to these models admits local physical excitations. Its degrees of freedom are defined on links and plaquettes, while gauge transformations are based on vertices and links of the underlying lattice. We specify the Hilbert space, define basic observables (including the Hamiltonian) and initiate a discussion on the model's phase diagram. The constructed model reduces in appropriate limits to topological theories with symmetries described by groups and crossed modules, lattice Yang-Mills theory and 2-form electrodynamics. We conclude by reviewing classifying spaces of crossed modules, with an emphasis on the direct relation between their geometry and properties of gauge theories under consideration.

Keywords: Gauge Symmetry, Lattice Quantum Field Theory, Topological Field Theories, Topological States of Matter

ArXiv ePrint: 2010.00888

https://doi.org/10.1007/JHEP03(2021)282

Contents

1 Introduction and summary

One of the most fruitful ideas in the study of phase transitions is Landau's theory [1], which classifies phases of matter according to their symmetries. Despite this success, it is currently known [2] that there exist transitions not driven by spontaneous symmetry breaking. In the case of gapped quantum systems, possibly with no symmetries, it has been proposed [3] that phases may be distinguished by their topological orders, which were later interpreted in more physical terms [4] as patterns of long range entanglement.

Topological aspects of many body quantum physics also turned out to play a role in understanding the quantum Hall effect [5], topological insulators [6], superconductors [7] and other quantum phases of matter [8, 9]. Several interesting applications arise in the study of geometry of Fermi surfaces [10]. Topologically nontrivial observables are often robust against local perturbations, and hence have been suggested to possess potential to be used in fault-tolerant quantum computation [11, 12].

A popular framework for description of topological order is that of Topological Quantum Field Theories (TQFTs) [13, 14]. Many well-known TQFTs are gauge theories, for example Chern-Simons [14], BF [15] or Dijkgraaf-Witten [16] theories. Several constructions, such as the Turaev-Viro [17] or Crane-Yetter [18] models, are based on quantum algebra, e.g. fusion categories. There is a closely related line of study in which gapped lattice hamiltonian models are considered, such as in the Kitaev's quantum double model [11], Levin-Wen string nets [19] or Walker-Wang model [20]. One of advantages of this approach is that it provides not only the space of ground states, but also the part of information about its possible excitations which is universal for the given gapped infrared renormalization group fixed point.

In this work we study a generalized lattice gauge theory, which may be seen as a non-topological extension of the Yetter's 2-type TQFT [21]. It is shown that various TQFTs, as well as the lattice Yang-Mills theory with finite gauge group [22–24] and 2-form gauge theory may be obtained as limits of our model. We emphasize that the Hamiltonian of our model is not the sum of commuting local terms, so dynamics of its local excitations is expected to be nontrivial and beyond the scope of description purely in terms of TQFT.

A crucial role in the analysis of physical systems is played by symmetries. It has been suggested [25, 26] that so-called higher symmetries, which act on extended objects, also play a significant role. An excellent example is provided by the center symmetry [27] in Yang-Mills theory (possibly with adjoint matter), which acts trivially on all local operators, but changes the value of Polyakov loops. Spontaneous breaking of this symmetry is responsible for a phase transition, which, however, is absent in QCD due to explicit breaking of the center symmetry.

Just as ordinary symmetries, higher symmetries may be used to derive selection rules on correlation functions. Moreover, they may be anomalous [28], which can be used to obtain theoretical constraints on the renormalization group flow. This is also related to the proposal [29] of Symmetry Protected Topological (SPT) phases [30] involving higher symmetries.

Higher symmetries may also be gauged, which leads to so-called higher gauge theories. One of the first models of this type, involving parallel transports over surfaces, was proposed in the context of string theory by Kalb and Ramond [31]. It has been argued [32] that higher gauge theories are necessarily abelian, essentially because there is no meaningful notion of time ordering on objects of dimension higher than one. To some extent this conviction is defied by models inspired by higher category theory [33–35]. In this case parallel transports are indeed valued in an abelian group, but they are defined in terms of genuinely non-abelian degrees of freedom. Besides truly dynamical models, higher gauge fields appear also in TQFTs such as the Yetter's model [21], its generalizations [29, 36–39] and hamiltonian formulations [40–42]. We refer to [40] for a comparison between these models. Higher gauge theories have also been proposed [29, 43, 44] as effective descriptions of Yang-Mills theory vacua.

Conventional gauge theories depend on a choice of a gauge group. It has been proposed [33] that generalization of this notion suitable for theories with transports over surfaces [45–47] is a 2-group. There are several equivalent ways to define these objects [48]. Here we choose to work with the formulation through crossed modules, whose definition will be recalled in the main text. We remark that these objects have also found applications in the classification of defects [49], mathematically modeled as solitonic sectors of sigma models.

Models discussed in this paper allow non-abelian degrees of freedom associated to edges and faces of a spatial lattice. They are subject to a constraint called fake flatness. This enables to consistently define parallel transports over spheres, besides the more standard Wilson loops. As usually, there is a gauge freedom, which is however reduced with respect to that present in Yetter's model. This is necessary in order to preserve the dynamical (rather than purely topological) nature of the 1-form gauge field, and hence to construct models generalizing the Yang-Mills theory. We propose a suitable hamiltonian and discuss its symmetries and various special cases, including Yetter's theory. Ground states are described in several integrable limits, which allows to formulate initial conjectures concerning the phase diagram.

The organization of this paper is as follows. Section 2 sets the stage for subsequent developments. Most of the contained material is not new and has been discussed for instance in [42], but we do hope that our way of presenting it may be useful for some readers. In subsection 2.1 we review basic geometric notions used in the text. This allows to state precisely what is meant by field configurations valued in a crossed module. Interpretation of these fields is discussed in subsection 2.2, where we define also the basic observables. Subsection 2.3 is devoted to transformations of the configuration space, including a presentation of our motivation to restrict the group of gauge transformations. To our knowledge, plaquette transformations introduced there have not appeared in the literature. This simple definition is important in the construction of 2-form electric operators. Examples in subsection 2.4 illustrate several aspects of the subtle interplay between spatial topology and algebra of crossed modules, in which the fields are valued. In section 3 we complete the construction of our model and present first results about its dynamics. Then in subsection 3.1 we specify the Hilbert space and define basic operators, including the hamiltonian. In order to make this more concrete, in subsection 3.2 we carry out the construction explicitly in the case of a hypercubic lattice and a particular crossed module. Symmetries of proposed hamiltonians are discussed in subsection 3.3. Afterwards, in subsection 3.4, we describe ground states of four integrable limits of our model and in each case relate it to some well-known TQFT. This is followed by subsection 3.5, in which it is shown that in a certain region of the phase diagram, intermediate between TQFTs and the full model, one finds Yang-Mills theory or 2-form gauge theory. Appendices A and B are devoted to a review of certain technical, albeit standard mathematical tools used in the main text. The more extensive appendix C is devoted to a discussion of classifying spaces of crossed modules. Relation of classifying spaces to gauge theories based on crossed modules is derived. This offers an interesting perspective on several properties of higher gauge theories. These results are known, but we are not aware of a similar exposition in the literature.

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A natural next step would be to analyze the dynamics of proposed models in more detail, e.g. using perturbation theory. There exists a natural candidate for a state sum formulation of the model presented here, which could be studied using strong coupling expansion or Monte Carlo methods. Similar questions may also be asked about corresponding models with continuous spacetimes and crossed modules of Lie groups.

2 Basic notions

2.1 Geometric setup and field configurations

Homotopy classes of (parametrized) paths in a topological space form a structure very similar to a group, since they can be composed in a way which is associative and admits multiplicative inverses. There is only one complication: composition $\gamma'\gamma$ exists only if the "source" of γ' coincides with the "target" of γ. This is abstracted by the notion of a **groupoid**, whose definition we now recall. A groupoid consists of:

1. sets G and Ob_G, called the set of arrows and the set of objects, respectively,
2. functions $s, t : G \to \mathrm{Ob}_G$, called the source and the target map,
3. an associative binary operation on G, denoted by juxtaposition, with $\gamma'\gamma$ defined if $\gamma, \gamma' \in G$ are such that $s(\gamma') = t(\gamma)$.

These data are subject to two axioms:

a) For every object x there exists an arrow id_x, with source and target x, such that $\gamma\,\mathrm{id}_x = \gamma$ and $\mathrm{id}_x\gamma = \gamma$ whenever these compositions are defined.

b) For every arrow γ there exists an arrow γ^{-1} with $s(\gamma^{-1}) = t(\gamma)$, $t(\gamma^{-1}) = s(\gamma)$, $\gamma^{-1}\gamma = \mathrm{id}_{s(\gamma)}$ and $\gamma\gamma^{-1} = \mathrm{id}_{t(\gamma)}$.

In further discussion we will abuse the language by calling the set G itself a groupoid.[1]

We note that for any $x \in \mathrm{Ob}_G$ the set of all $\gamma \in G$ with $x = t(\gamma) = s(\gamma)$ is a group. In particular, if Ob_G has exactly one element, then G itself is a group.

If B is a subspace of a topological space A, the **fundamental groupoid** $\pi_1(A; B)$ has B as its set of objects and the set of homotopy classes of paths in A with (fixed) endpoints in B as the set of arrows. Source and target maps are obvious. A composition $\gamma\gamma'$ is defined as γ' followed by γ, which makes sense if $s(\gamma) = t(\gamma')$. We note that the fundamental group $\pi_1(A; b)$ of A based at $b \in B$ may be described as $\{\gamma \in \pi_1(A; B) \,|\, t(\gamma) = s(\gamma) = b\}$.

[1]Identity arrows id_x are often regarded as a part of the data defining a groupoid. However, since they are necessarily unique, we prefer to include their existence among the axioms a), b) rather than among groupoid's data 1. – 3.

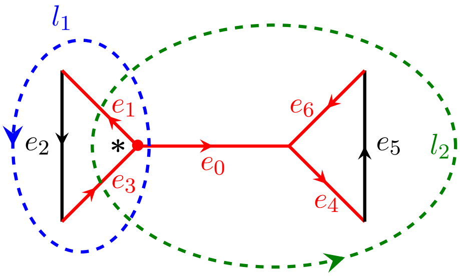

Figure 1. Illustration of possible sets of generators of $\pi_1(X_1; X_0)$ for a certain space X. Edges are depictured by continuous lines, with a maximal tree distinguished by the red color. Two independent loops based at the point $*$ (the big dot) are indicated by dashed lines.

In our applications we shall consider connected spaces X equipped with a lattice decomposition.[2] In this situation we have a chain of inclusions

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_d = X, \tag{2.1}$$

where d is the dimension of X. Here X_0 is the set of vertices (also called lattice sites or 0-cells), X_1 is constructed by gluing in edges (links or 1-cells) to X_0, X_2 by gluing in faces (plaquettes or 2-cells) to X_1 etc. 3-cells will be referred to as balls. We will make an extensive use of the groupoid $\pi_1(X_1; X_0)$. Its set of arrows may be described as the free groupoid generated by the edges of X. This means that any arrow is a product of some number of edges (identity arrows being understood as empty products), and that the only relations between two such products are those which follow from associativity of composition and identification of an edge e with orientation reversed with the inverse of the edge e.

The above description of the fundamental groupoid is convenient for applications in field theory, since it is given in terms of local data. In some arguments another set of generators proves to be useful. Let us choose some $* \in X_0$. The fundamental group $\pi_1(X_1; *)$ is free [50, p. 83], i.e. there exists a set L of generators satisfying no non-trivial relations, called a basis of loops. Secondly, we may choose a maximal tree T, i.e. a maximal set of edges with the property that there exists no non-trivial loop composed entirely of edges in T. Then $\pi_1(X_1; X_0)$ is freely generated by $L \cup T$.

A simple example is in order. Consider the space illustrated on figure 1. It has seven edges $\{e_i\}_{i=0}^{6}$. A basis of loops based at $*$ may be taken as $L = \{l_1, l_2\}$, where $l_1 = e_3 e_2 e_1$ and $l_2 = e_0^{-1} e_6 e_5 e_4 e_0$. Set $T = \{e_0, e_1, e_3, e_4, e_6\}$ is a maximal tree. groupoid $\pi_1(X_1; X_0)$ is generated by the loops l_1, l_2 and the edges in T. There are no non-trivial relations between these generators.

Now let G, G' be groupoids. Map $F : G \to G'$ is called a homomorphism if:

1. there exists a map $F_0 : \mathrm{Ob}_G \to \mathrm{Ob}_{G'}$ such that $s \circ F = F_0 \circ s$ and $t \circ F = F_0 \circ t$,

2. $F(\gamma\gamma') = F(\gamma)F(\gamma')$ whenever $s(\gamma) = t(\gamma')$.

[2] For the most part it would be sufficient to consider smooth manifolds with a triangulation, though we prefer to allow more general decompositions: CW-complexes with cellular attaching maps for 2-cells and 3-cells. Here we have in mind the standard CW-decompositions of S^1 and S^2 with exactly two cells. Many results can be formulated in even larger generality, but this class of topological spaces is sufficient for our purposes.

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We note that F_0 is uniquely determined by F and that the first property guarantees that the second one makes sense. The second property together with existence of inverse arrows implies that F takes identity arrows to identity arrows. If G, G' are groups, F is simply a homomorphism of groups.

To give a concrete example: lattice gauge field on X valued in a group G may be defined as a homomorphism $\pi_1(X_1; X_0) \to G$. Since there are no relations between distinct edges, regarded as arrows of $\pi_1(X_1; X_0)$, defining a lattice gauge field amounts to specifying independently a group element $g_e \in G$ for every edge e. The element associated to a path $\gamma = e_n \dots e_1$ is $g_\gamma = g_{e_n} \dots g_{e_1}$. Alternatively, a lattice gauge field may be specified by giving a homomorphism $\pi_1(X_1; *) \to G$ for some $* \in X_0$ and the values of g_e for edges e from any maximal tree T. These data can be chosen independently because there are no relations between generators of $\pi_1(X_1; *)$ and elements of T. In order to capture two-dimensional aspects of geometry needed to formulate models considered in this work, we need to review another algebraic structure. A **crossed module** of groupoids is a quadruple $(G, H, \triangleright, \partial)$ consisting of:

1. groupoids G, H with $\mathrm{Ob}_H = \mathrm{Ob}_G$ and $s(h) = t(h)$ for any[3] $h \in H$,
2. homomorphism $\partial : H \to G$ with $\partial_0 : \mathrm{Ob}_H \to \mathrm{Ob}_G$ the identity map,
3. action $\triangleright$ of G on H: $g \triangleright h \in H$ with $t(g \triangleright h) = t(g)$ is defined for $g \in G$, $h \in G$ whenever $s(g) = t(h)$.

These data are subject to the axioms:

1. $\mathrm{id}_{t(h)} \triangleright h = h$ for any $h \in H$,
2. $(gg') \triangleright h = g \triangleright (g' \triangleright h)$ whenever $s(g) = t(g')$ and $s(g') = t(h)$,
3. $g \triangleright (hh') = (g \triangleright h)(g \triangleright h')$ whenever $s(g) = t(h) = t(h')$,
4. **1st Peiffer identity:** $\partial(g \triangleright h) = g \partial(h) g^{-1}$ whenever $s(g) = t(h)$,
5. **2nd Peiffer identity:** $(\partial h) \triangleright h' = h h' h^{-1}$ whenever $t(h) = t(h')$.

Properties 1–3 characterize the action $\triangleright$, while Peiffer identites 4 and 5 are compatibility conditions between $\triangleright$ and ∂. If G has exactly one object, $(G, H, \partial, \triangleright)$ is called a crossed module of groups. We proceed to motivate this lengthy definition by giving the example most important for our models.

For a topological space A, its subspace B and an element $b \in B$, the second relative homotopy group $\pi_2(A, B; b)$ of A relative to B and base b is defined as the set of homotopy classes of maps $[0,1]^2 \to A$ such that $[0,1] \times \{1\}$ is mapped to B and $([0,1] \times \{0\}) \cup (\{0,1\} \times [0,1])$ is mapped to b. See figure 2 for a pictorial representation of these conditions.

Multiplication of elements of $\pi_2(A, B; b)$ is given by horizontal concatenation, see figure 3. One may also show [50, p. 343] that elements of $\pi_2(A, B; b)$ describe homotopy

[3] In other words, there are no arrows between different objects in H. A groupoid with this property is sometimes called totally disconnected or totally intransitive.

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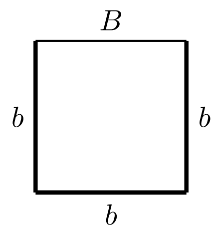

Figure 2. Illustration of conditions satisfied by maps representing elements of the second relative homotopy group $\pi_2(A, B; b)$.

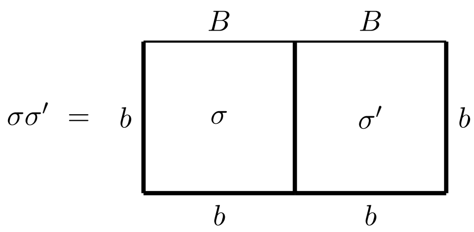

Figure 3. Definition of the product in $\pi_2(A, B; b)$, which is given by horizontal concatenation.

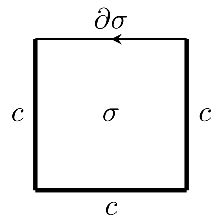

Figure 4. Definition of ∂: homotopy class of the map given by a square σ is mapped to the loop given by its upper edge.

classes of maps of a disc to A which map the boundary to B and a single point of the boundary to b. In the case $B = \{b\}$ we abbreviate $\pi_2(A, B; b) = \pi_2(A; b)$. Elements of this group are homotopy classes of maps $S^2 \to A$, since a square with its boundary crushed to a point is a two-sphere.

More generally, for a subspace $C \subseteq B$ we let $\pi_2(A, B; C)$ be the groupoid with object set C and the set of arrows from c to c' given by $\pi_2(A, B; c)$ if $c = c'$ and empty otherwise. Homomorphism $\partial : \pi_2(A, B; C) \to \pi_1(B; C)$ is defined by mapping the homotopy class of a map σ to the homotopy class of $\sigma|_{[0,1]\times\{1\}}$, see figure 4.

Last, but not least, an action of $\pi_1(B; C)$ on $\pi_2(A, B; C)$ is defined on figure 5.

Based upon the inspection of figures 3–5 one can show that $\Pi_2(A, B; C) = (\pi_1(B; C), \pi_2(A, B; C), \partial, \triangleright)$ satisfies all axioms of a crossed module of groupoids. For a detailed proof we refer to [60].

A homomorphism of crossed modules of groupoids $(G, H, \partial, \triangleright) \to (G', H', \partial', \triangleright')$ is a pair of homomorphisms of groupoids, $E : G \to G'$ and $F : H \to H'$, such that $\partial' \circ F = E \circ \partial$ and $F(g \triangleright h) = E(g) \triangleright' F(h)$ whenever $s(g) = t(h)$. Now let $\mathbb{G} = (\mathcal{E}, \Phi, \Delta, \triangleright)$ be a fixed crossed module of groups with finite $\mathcal{E}$ and Φ. A $\mathbb{G}$-valued lattice gauge field is defined

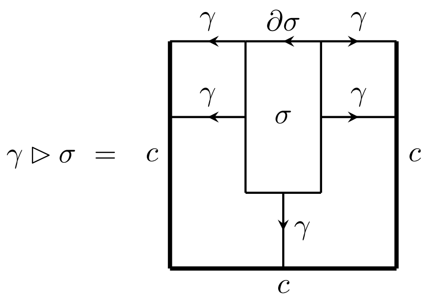

Figure 5. Definition of the action of $\pi_1(B;C)$ on $\pi_2(A,B;C)$. Here γ is a path from c' to c and σ belongs to $\pi_2(A,B;c')$.

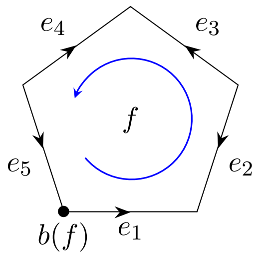

Figure 6. Pentagonal plaquette with a chosen orientation of edges and of the face. In this case $\partial f = e_5 e_4^{-1} e_3 e_2^{-1} e_1$.

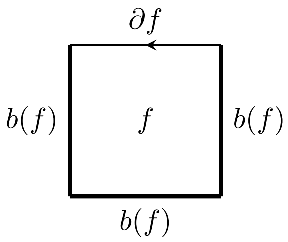

Figure 7. Schematic representation of a map representing the element in $\pi_2(X_2, X_1; b(f))$ corresponding to a plaquette f.

as a homomorphism $\Pi_2(X_2, X_1; X_0) \to \mathbb{G}$. In order to turn this concise definition into an operational one, we need a description of the groupoid $\pi_2(X_2, X_1; X_0)$ in terms of explicit generators, preferably constructed in terms of local data. It is a nontrivial fact, which follows from the results of Whitehead [51, 52], that this is indeed possible.[4] This is what we will review next.

For every face f we choose a basepoint $b(f)$ and an orientation. The boundary of f then forms a loop ∂f based at $b(f)$. See figure 6 for an example.

There exists a corresponding element $f \in \pi_2(X_2, X_1; b(f))$, given by the homotopy class of any map of the schematic form depictured on figure 7.

[4]The version of this theorem for spaces with many base points (which is actually the one we consider here) can be found in [60]. It can be easily deduced from the pointed version.

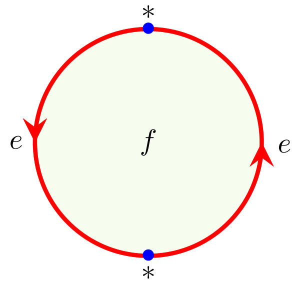

Figure 8. The real projective plane is a disc with antipodal points of the bounding circle identified. It can be constructed by attaching a plaquette to a circle along a map of winding number two.

By acting on faces with paths it is possible to obtain new elements, possibly based at different points. It turns out that the set of all $\gamma \triangleright f$ with $s(\gamma) = b(f)$ generates the groupoid $\pi_2(X_2, X_1; b(f))$. The only non-trivial relations between these elements follow from Peiffer identities and are of the form

$$(\gamma\, \partial f\, \gamma^{-1}) \triangleright (\gamma' \triangleright f') = (\gamma \triangleright f)\,(\gamma' \triangleright f')\,(\gamma \triangleright f)^{-1} \tag{2.2}$$

for every γ, γ', f and f' such that $t(\gamma') = t(\gamma)$, $s(\gamma') = b(f')$ and $s(\gamma) = b(f)$.

For the sake of example, we consider the real projective plane, $X = \mathbb{RP}^2$. It admits a decomposition with exactly one cell in every dimension up to 2 — see figure 8. In this case the groupoid $\pi_1(X_1; X_0)$ has one object $*$ and one generator e. There is one plaquette f, with $\partial f = e^2$. The relative homotopy group $\pi_2(X_2, X_1; *)$ is generated by elements $f_n := e^n \triangleright f$, $n \in \mathbb{Z}$. The first Peiffer identity gives $\partial f_n = e^2$, so relations (2.2) reduce to

$$f_{m+2} = f_n f_m f_n^{-1} \qquad \text{for all } n, m \in \mathbb{Z}. \tag{2.3}$$

Evaluating this for $m = n$ gives $f_n = f_{n+2}$. Thus all generators can be expressed in terms of f_0 and f_1. Secondly, taking $n = 0$, $m = 1$ gives $f_0 f_1 = f_1 f_0$. There are no other independent relations, so $\pi_2(X_2, X_1; *) \cong \mathbb{Z}^2$.

We are now ready to explain what are field configurations in the considered models. In order to define a homomorphism $\Pi_2(X_2, X_1; X_0) \to \mathbb{G}$ we have to assign an element $\epsilon_e \in \mathcal{E}$ to every edge e and $\varphi_f \in \Phi$ to every face f. Since this assignment is to define a homomorphism $\pi_1(X_1; X_0) \to \mathcal{E}$, we map a path $\gamma = e_n \dots e_1$ to $\epsilon_\gamma = \epsilon_{e_n} \dots \epsilon_{e_1}$. Element $\gamma \triangleright f \in \pi_2(X_2, X_1; t(\gamma))$ has to be sent to $\epsilon_\gamma \triangleright \varphi_f$, by the definition of a homomorphism of crossed modules. Since any arrow in $\pi_2(X_2, X_1; X_0)$ is a product of arrows of this form, the element $\varphi_\sigma \in \Phi$ assigned to any σ is determined. We still have to make sure that this is consistent. Firstly, the definition of a homomorphism asserts that we should have

$$\Delta \varphi_f = \epsilon_{\partial f} \qquad \text{for any face } f. \tag{2.4}$$

This is a non-trivial constraint on the collections $\boldsymbol{\epsilon} = \{\epsilon_e\}$, $\boldsymbol{\varphi} = \{\varphi_f\}$, called **fake flatness**. We claim that there are no other constraints, since compatibility with the relation (2.2) is automatic. Indeed, equality

$$(\epsilon_\gamma\, \Delta\varphi_f\, \epsilon_\gamma^{-1}\, \epsilon_{\gamma'}) \triangleright \varphi_{f'} = (\epsilon_\gamma \triangleright \varphi_f)\,(\epsilon_{\gamma'} \triangleright \varphi_{f'})\,(\epsilon_\gamma \triangleright \varphi_f)^{-1} \tag{2.5}$$

follows from the fake flatness condition and Peiffer identities in $\mathbb{G}$.

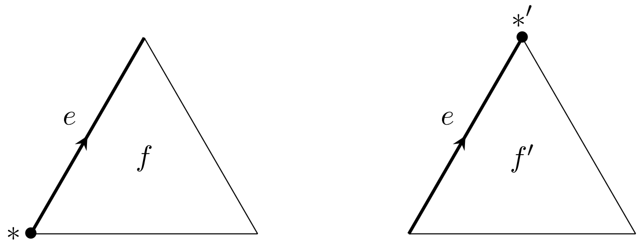

Figure 9. Illustration of the change of a base point. In this case $f' = e \triangleright f$.

To understand how field configurations look like in practice, consider the example of X taken to be the pentagon presented on figure 6. A field configuration consists of elements $\epsilon_{e_1}, \ldots, \epsilon_{e_5} \in \mathcal{E}$ and $\varphi_f \in \Phi$ subject to the constraint

$$\Delta\varphi_f = \epsilon_{e_5}\epsilon_{e_4}^{-1}\epsilon_{e_3}\epsilon_{e_2}^{-1}\epsilon_{e_1}. \tag{2.6}$$

In the above discussion we have been forced to choose base points and orientations for the elementary plaquettes. Distinct choices correspond to distinct choices of generators of the same algebraic structure. We close this section with an explanation how generators are transformed upon a change of these choices:

1. Change of orientation of a plaquette maps the element f to f^{-1}.

2. Change of the base point from $*$ to $*'$ (with both elements belonging to the boundary of f) changes f to $\gamma \triangleright f$, where γ is a path from $*$ to $*'$ along the boundary of f. For an example see figure 9. If f is simply-connected, the element $\gamma \triangleright f$ does not depend on the choice of γ. Indeed, in this case any other allowed path takes the form $\gamma' = \gamma\,(\partial f)^n$ for some n and $\partial f \triangleright f = f$, by the second Peiffer identity.

The above discussion is concerned with generators of an abstract group describing the geometry. Corresponding transformation laws for field configurations are of the form $\varphi_f \mapsto \varphi_f^{-1}$ and $\varphi_f \mapsto \epsilon_\gamma \triangleright \varphi_f$ for points 1. and 2., respectively. The fact that $\epsilon_\gamma \triangleright \varphi_f$ is then independent of the choice of γ relies on the fake flatness constraint.

2.2 Degrees of freedom and holonomies

In models based on crossed modules there are, besides holonomies along loops (built out of degrees of freedom located on edges), also holonomies along surfaces (built out of degrees of freedom located on edges and faces). In order to explain their construction we first need to discuss certain basic properties of crossed modules.

Let $(G, H, \partial, \triangleright)$ be a crossed module of groups. We note two important consequences of the first Peiffer identity:

1. The image $\mathrm{im}(\partial)$ of ∂ is a normal subgroup of G. Thus there is a group structure on the quotient space $\mathrm{coker}(\partial) = G/\mathrm{im}(\partial)$.

2. The action of G on H preserves $\ker(\partial)$, the kernel of ∂:

$$h \in \ker(\partial) \implies \forall g \in G \quad g \triangleright h \in \ker(\partial), \tag{2.7}$$

and two conclusions from the second Peiffer identity:

1. $\ker(\partial)$ is a central subgroup of H. In particular $\ker(\partial)$ is abelian.

2. Elements of $\mathrm{im}(\partial)$ act trivially on $\ker(\partial)$, i.e. $g \triangleright h = h$ for all $g \in \mathrm{im}(\partial)$ and $h \in \ker \partial$. Thus there is an induced action of the group $\mathrm{coker}(\partial)$ on $\ker(\partial)$.

Furthermore, if $(E, F) : (G, H, \partial, \triangleright) \to (G', H', \partial', \triangleright')$ is a homomorphism of crossed modules of groups, then:

1. $E(\mathrm{im}(\partial)) \subseteq \mathrm{im}(\partial')$, so there is an induced map $\overline{E} : \mathrm{coker}(\partial) \to \mathrm{coker}(\partial')$.

2. F maps $\ker(\partial)$ to $\ker(\partial')$. We denote the induced map $\ker(\partial) \to \ker(\partial')$ by $\overline{F}$.

For future use we remark that if $\overline{E}$ and $\overline{F}$ are group isomorphisms, (E, F) is said to be a **weak isomorphism**. Existence of a weak isomorphism $\mathbb{G} \to \mathbb{G}'$ does not imply[5] that there is a weak isomorphism $\mathbb{G}' \to \mathbb{G}$. Thus in order for this notion to yield an equivalence relation, one declares two crossed modules $\mathbb{G}$ and $\mathbb{G}'$ to be **weakly equivalent** if there exist a family of crossed modules $\mathbb{G}_1, \dots, \mathbb{G}_n$ and a zig-zag sequence of weak isomorphisms of the form

$$\mathbb{G} \longrightarrow \mathbb{G}_1 \longleftarrow \mathbb{G}_2 \longrightarrow \dots \longleftarrow \mathbb{G}_n \longrightarrow \mathbb{G}'. \tag{2.8}$$

In other words, weak equivalence is the coarsest equivalence relation such that weakly isomorphic crossed modules are equivalent.

Let us now specialize to the crossed module $\Pi_2(X_2, X_1; *)$ for some $* \in X_0$. As reviewed in the appendix A, $\mathrm{coker}(\partial)$ and $\ker(\partial)$ are the fundamental group of X and the second homotopy group of X_2, respectively. Therefore any field configuration induces homomorphisms $\pi_1(X; *) \to \mathrm{coker}(\Delta)$ and $\pi_2(X_2; *) \to \ker(\Delta)$. We will now explain their significance.

Consider a field configuration given by $\boldsymbol{\epsilon}$ and $\boldsymbol{\varphi}$. Element $\epsilon_\gamma \in \mathcal{E}$ assigned to a path γ has the interpretation of a parallel transport from $s(\gamma)$ to $t(\gamma)$ along γ. Parallel transports along closed paths ($s(\gamma) = t(\gamma)$) will be called **1-holonomies**, to distinguish them from **2-holonomies**, to be considered soon.

We define $\overline{\epsilon}_\gamma$ as the reduction of ϵ_γ modulo $\mathrm{im}(\Delta)$. Assignment $\gamma \mapsto \overline{\epsilon}_\gamma$ defines an ordinary $\mathrm{coker}(\Delta)$-valued lattice gauge field $\overline{\boldsymbol{\epsilon}}$. Its definition is motivated by inspecting the fake flatness condition (2.4) reduced modulo $\mathrm{im}(\Delta)$:

$$\overline{\epsilon}_{\partial f} = 1 \qquad \text{for any face } f. \tag{2.9}$$

[5]For the sake of example, consider the crossed module with $H = \mathbb{Z}_2$ and trivial G, ∂ and $\triangleright$. Secondly, let us take $H' = \mathbb{Z}_4$, $G' = \mathbb{Z}_2$, ∂' given by reduction modulo two and trivial $\triangleright'$. Standard embedding $\mathbb{Z}_2 \to \mathbb{Z}_4$, together with the trivial homomorphism $G \to G'$, is a weak isomorphism. It is easy to check that there exists no weak isomorphism in the opposite direction.

This is the statement that $\overline{\boldsymbol{\epsilon}} = \{\overline{\epsilon}_e\}$ is a flat gauge field: the holonomy along any loop in X which bounds a surface is trivial,[6] so holonomies along homotopic loops are equal. In other words, $\overline{\boldsymbol{\epsilon}}$ defines a homomorphism $\pi_1(X; X_0) \to \mathrm{coker}(\Delta)$.

To further understand the fake flatness condition, consider the problem of finding its solutions $\boldsymbol{\epsilon}$, $\boldsymbol{\varphi}$ for a fixed flat $\overline{\boldsymbol{\epsilon}}$. First, each ϵ_e is determined by $\overline{\epsilon}_e$ up to multiplication by $\Delta\psi_e$ for some $\psi_e \in \Phi$. Having chosen any particular $\boldsymbol{\epsilon}$, we are guaranteed by flatness of $\overline{\boldsymbol{\epsilon}}$ that each $\epsilon_{\partial f}$ belongs to $\mathrm{im}(\Delta)$: there exists some $\varphi_f \in \Phi$, unique up to multiplication by any $\chi_f \in \ker(\Delta)$, such that $\Delta\varphi_f = \epsilon_{\partial f}$.

The above discussion may be summarized as follows. Gauge field valued in a crossed module may be though of as consisting of three components:

1. $\mathrm{coker}(\Delta)$-valued field located on edges, constrained by (2.9) and hence defining a flat gauge field $\overline{\boldsymbol{\epsilon}}$,

2. $\mathrm{im}(\Delta)$-valued degrees of freedom located on edges, responsible for the freedom in the choice of $\boldsymbol{\epsilon}$ for a given $\overline{\boldsymbol{\epsilon}}$,

3. $\ker(\Delta)$-valued degrees of freedom located on faces, responsible for the freedom in the choice of $\boldsymbol{\varphi}$ for a given $\boldsymbol{\epsilon}$.

As in ordinary gauge theory, some degrees of freedom are eventually removed by introducing a "gauge equivalence" relation on the set of field configurations. This will be discussed in subsection 2.3.

Typical observables sensitive to degrees of freedom of the third type are (functions of) 2-holonomies, i.e. elements $\varphi_\sigma \in \Phi$ assigned to $\sigma \in \ker(\partial)$. Notice that φ_σ are automatically in $\ker(\Delta)$. Indeed, $\Delta\varphi_\sigma = \varphi_{\partial\sigma} = 1$. This gives the promised homomorphism $\pi_2(X_2; *) \to \ker(\Delta)$. It may be interpreted as a two-dimensional analogue of parallel transport along closed paths, with loops replaced by spheres embedded in X_2.

To summarize the above discussion, $\ker(\Delta)$-valued holonomy along any sphere in X_2 is defined. We will now demonstrate how to compute it in some simple examples.

Consider the triangulation of a two-sphere presented on figure 10. We choose $*$ as the base point of f_1, f_2, f_3 and $t(e_1)$ as the base point of f_4. Faces $f_1, \dots, f_4$ are oriented so that

$$\partial f_1 = e_2^{-1} e_4^{-1} e_1, \quad \partial f_2 = e_3^{-1} e_5^{-1} e_2, \quad \partial f_3 = e_1^{-1} e_6 e_3, \quad \partial f_4 = e_4 e_5 e_6^{-1}. \tag{2.10}$$

It is well known that the second homotopy group of S^2 is infinite cyclic. Choice of one of two possible generators of this group is equivalent to a choice of orientation. One may construct a generator by multiplying the four faces (all transported to a common base point by acting with edges) in such a way that an element with trivial boundary is obtained. There is more than one way to do this, as shown on figure 11. It is not difficult to convince oneself that the two elements σ, σ' presented on figure 11 represent the same

[6] This follows directly from the condition above only for surfaces in X_2. Here we are also using the fact that attaching 3-cells to a space does not change its fundamental group [50, Prop. 1.26(b)].

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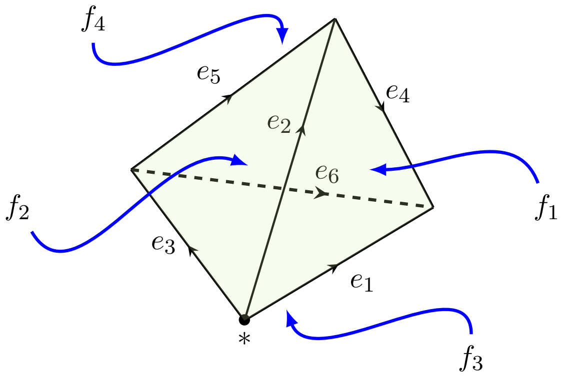

Figure 10. Tetrahedron as an example of a triangulation of a 2-sphere. Chosen orientations of edges are indicated by arrows.

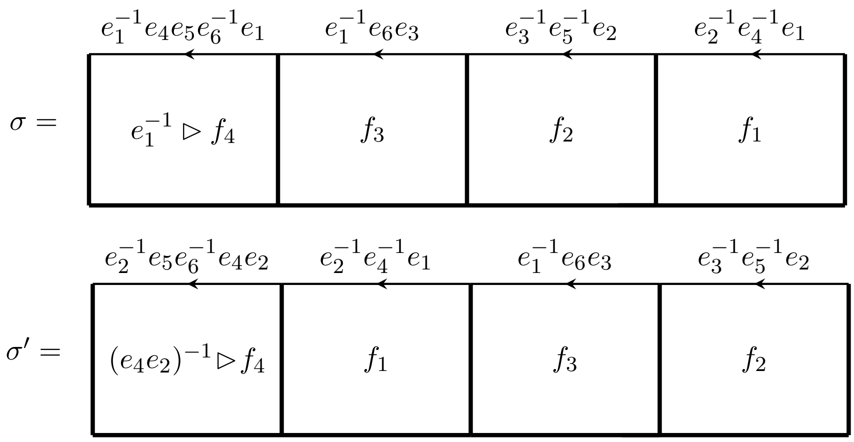

Figure 11. Graphical representation of two ways to construct a generator of $\pi_2(X_2, X_1; *)$ for the tetrahedron from figure 10: $\sigma = (e_1^{-1} \triangleright f_4)f_3f_2f_1$ and $\sigma' = ((e_4e_2)^{-1} \triangleright f_4)f_1f_3f_2$. In fact we have $\sigma = \sigma'$.

orientation. Thus they must be equal. We will now check this by a direct computation:

$$\sigma' = \left(\underbrace{(e_4e_2)^{-1}e_1}_{\partial f_1} e_1^{-1} \triangleright f_4\right) f_1f_3f_2 = f_1(e_1^{-1} \triangleright f_4)f_3f_2 = f_1\sigma f_1^{-1} = \sigma, \tag{2.11}$$

where we applied the second Peiffer identity, inserted the definition of σ and used the fact that σ is central in the second, third and fourth equalities, respectively.

Having constructed the element $\sigma \in \pi_2(X_2, X_1; *)$, we compute φ_σ simply by replacing each edge e (resp. plaquette f) in the definition of σ by the corresponding ϵ_e (resp. φ_f). Thus (compare with figure 11):

$$\varphi_\sigma = (\epsilon_{e_1}^{-1} \triangleright \varphi_{f_4})\, \varphi_{f_3}\, \varphi_{f_2}\, \varphi_{f_1}. \tag{2.12}$$

By construction, φ_σ may also be computed as $((\epsilon_{e_4}\, \epsilon_{e_2})^{-1} \triangleright \varphi_{f_4})\, \varphi_{f_1}\, \varphi_{f_3}\, \varphi_{f_2}$.

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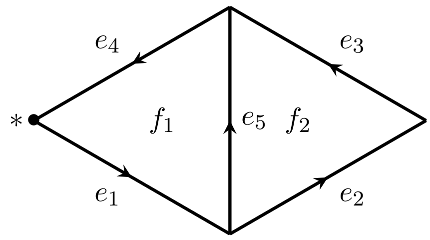

Figure 12. Triangulation of a disc. We take $*$ and $s(e_2)$ as the base points of f_1 and f_2, respectively. Both faces are oriented counterclockwise.

More generally, element σ may always be constructed in an essentially unique way for any decomposition of a two-sphere with a chosen base point and orientation (possibly embedded in a larger space). The case particularly important for us is that of a sphere bounding a ball q, oriented and based at a point $b(q) \in X_0$. In this case we denote the corresponding element $\sigma \in \pi_2(X_2; b(q))$ by ∂q.

We close this section with remarks about $\Pi_2(X_2, X_1; X_0)$ in the case when X is (a decomposition of) a disc. Then the first and the second (non-relative) homotopy groups are trivial. Thus ∂ has trivial kernel and cokernel, i.e. it is an isomorphism. This means that a polygon embedded in X bounded by a loop l corresponds to the uniquely determined element $\partial^{-1}(l) \in \pi_2(X_2, X_1; s(l))$. There is more than one way to construct the element $\partial^{-1}(l)$ out of elementary plaquettes, but they are all equal due to Peiffer identities. Of course there is still some arbitrariness in the choice of the base point $s(l)$, but groups $\pi_2(X_2, X_1; x)$ with distinct $x \in X_0$ are canonically isomorphic. Remarks of this paragraph are also applicable to calculations performed in completely general geometries X, as long as only elements constructed out of edges and plaquettes in a contractible subcomplex of X_2 are involved.

As an example, let us consider the triangulation of a disc depictured on the figure 12. With the chosen base points and orientations of faces we have

$$\partial f_1 = e_4 e_5 e_1, \qquad \partial f_2 = e_5^{-1} e_3 e_2. \tag{2.13}$$

We will construct the element corresponding to the whole disc out of elementary plaquettes and edges. We choose the counterclockwise orientation and pick $*$ as the base point. Then the bounding loop is $l = e_4 e_3 e_2 e_1$. We observe that

$$l = \underbrace{e_4 e_5 e_1}_{\partial f_1} e_1^{-1} \underbrace{e_5^{-1} e_3 e_2}_{\partial f_2} e_1 = \partial \left(f_1 \left(e_1^{-1} \rhd f_2 \right) \right), \tag{2.14}$$

so $\partial^{-1}(l) = f_1 \left(e_1^{-1} \rhd f_2 \right)$. On the other hand, we also have $l = \partial \left(\left(e_4 e_5 \rhd f_2 \right) f_1 \right)$. Thus $f_1 \left(e_1^{-1} \rhd f_2 \right) = \left(e_4 e_5 \rhd f_2 \right) f_1$. Indeed, this is easy to verify directly:

$$f_1 \left(e_1^{-1} \rhd f_2 \right) = \underbrace{f_1 \left(e_1^{-1} \rhd f_2 \right) f_1^{-1}}_{\text{Peiffer}} f_1 = \left(\left(\partial f_1 e_1^{-1} \right) \rhd f_2 \right) f_1 = \left(e_4 e_5 \rhd f_2 \right) f_1. \tag{2.15}$$

2.3 Gauge and electric transformations

As in ordinary gauge theory, there exist two particularly important broad classes of transformations of the set of field configurations. Firstly, we have gauge transformations. They

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$$
\begin{tikzcd}
(\boldsymbol{\epsilon}, \boldsymbol{\varphi}) \arrow[d, "\{\xi_v\}"'] \arrow[dr, "\{\xi'_v\, \xi_v\}"] & \\
(\boldsymbol{\epsilon}', \boldsymbol{\varphi}') \arrow[r, "\{\xi'_v\}"] & (\boldsymbol{\epsilon}'', \boldsymbol{\varphi}'')
\end{tikzcd}
$$

Figure 13. Composition rule for vertex transformations: $\{\xi_v\}$ followed by $\{\xi'_v\}$ coincides with $\{\xi'_v\, \xi_v\}$.

describe redundancies in the description of the system, since configurations related by gauge transformations are regarded as physically indistinguishable. Secondly, there are transformations which are used to define higher analogues of the electric field operators in the quantized theory. Here we will discuss both types at the same time, as they are closely related.[7]

Another distinction between various transformation arises from geometric considerations: p-form transformations are parametrized by data associated to geometric objects of dimension p. Here we will consider vertex (0-form) transformations, regarded as gauge redundancies, analogous to those present in the ordinary gauge theory. Secondly, there will be edge (1-form) transformations. Declaring them to be gauge transformations is necessary to obtain the Yetter's topological field theory and its twisted versions. We will discuss the possibility to restrict the group of gauge transformations. This increases the number of physical degrees of freedom and hence allows to construct models with richer dynamics. Finally, we will introduce plaquette (2-form) transformations. They play the role of electric transformations and are very analogous to corresponding transformations in abelian 2-form gauge theory.

We begin with the discussion of **vertex transformations**. They are parametrized by collections $\boldsymbol{\xi} = \{\xi_v\}$ of elements of $\mathcal{E}$ indexed by lattice sites. Their action on $\boldsymbol{\epsilon}$ is as for usual gauge fields, while φ_f transforms as a matter field placed on the lattice site $b(f)$:

$$\epsilon'_e = \xi_{t(e)}\, \epsilon_e\, \xi_{s(e)}^{-1}, \qquad \varphi'_f = \xi_{b(f)} \rhd \varphi_f. \tag{2.16}$$

We will call them vertex transformations. They preserve the fake flatness, since

$$\Delta\varphi'_f = \Delta\left(\xi_{b(f)} \rhd \varphi_f\right) \overset{\text{Peiffer}}{=} \xi_{b(f)}\, \Delta\varphi_f\, \xi_{b(f)}^{-1} \overset{\text{f.f.}}{=} \xi_{b(f)}\, \epsilon_{\partial f}\, \xi_{b(f)}^{-1} = \epsilon'_{\partial f}. \tag{2.17}$$

Thus they define a left action of the group $\mathcal{E}_X^{(0)}$ of all collections $\boldsymbol{\xi}$ (with vertex-wise multiplication, see figure 13) on the set of field configurations. All transformations in this group will be regarded as gauge redundancies.

Secondly, an **edge transformation** is parametrized by a collection $\boldsymbol{\psi} = \{\psi_e\}$ of elements of Φ. It changes $\boldsymbol{\epsilon}$ according to

$$\epsilon'_e = \Delta\psi_e\, \epsilon_e. \tag{2.18}$$

[7]One example of this fact is that in conventional gauge theory quantized in temporal gauge, time-independent gauge transformations are generated by the divergence of the electric field. Secondly, the center 1-form symmetry operators are also of electric type: they shift the gauge field by a center-valued cocycle. This operation reduces to a gauge transformation for cocycles of trivial cohomology class.

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Before we give the transformation law for $\boldsymbol{\varphi}$, let us inspect how ϵ_γ changes for general γ. We observe that $\bar{\epsilon}'_e = \bar{\epsilon}_e$ implies that $\bar{\epsilon}'_\gamma = \bar{\epsilon}_\gamma$ for any path γ. Thus

$$\epsilon'_\gamma = \Delta\psi^{(\epsilon)}_\gamma \, \epsilon_\gamma \tag{2.19}$$

for some $\psi^{(\epsilon)}_\gamma$, which depends on $\boldsymbol{\psi}$ as well as on $\boldsymbol{\epsilon}$. This equation determines each $\psi^{(\epsilon)}_\gamma$ up to multiplication by an element of $\ker(\Delta)$. As a step towards an unambiguous definition, we consider a composite path $\gamma\gamma'$ and evaluate $\epsilon'_{\gamma\gamma'}$ in two different ways. Firstly, by equation (2.19), it is equal to $\Delta\psi^{(\epsilon)}_{\gamma\gamma'}\,\epsilon_{\gamma\gamma'}$. On the other hand we have $\epsilon'_{\gamma\gamma'} = \epsilon'_\gamma\,\epsilon'_{\gamma'}$. Applying (2.19) to the two terms separately we obtain

$$\epsilon'_{\gamma\gamma'} = \Delta\psi^{(\epsilon)}_\gamma \underbrace{\epsilon_\gamma\,\Delta\psi^{(\epsilon)}_{\gamma'}\,\epsilon_\gamma^{-1}}_{\text{Peiffer}} \epsilon_\gamma\,\epsilon_{\gamma'} = \Delta\left(\psi^{(\epsilon)}_\gamma \left(\epsilon_\gamma \rhd \psi^{(\epsilon)}_{\gamma'}\right)\right)\epsilon_{\gamma\gamma'}. \tag{2.20}$$

Comparison of the two results yields

$$\Delta\psi^{(\epsilon)}_{\gamma\gamma'}\,\epsilon_{\gamma\gamma'} = \Delta\left(\psi^{(\epsilon)}_\gamma \left(\epsilon_\gamma \rhd \psi^{(\epsilon)}_{\gamma'}\right)\right)\epsilon_{\gamma\gamma'}. \tag{2.21}$$

This formula has the consequence that, perhaps up to multiplication of the right hand side by an element of $\ker(\Delta)$,

$$\psi^{(\epsilon)}_{\gamma\gamma'} = \psi^{(\epsilon)}_\gamma \left(\epsilon_\gamma \rhd \psi^{(\epsilon)}_{\gamma'}\right). \tag{2.22}$$

It is convenient to *define* $\psi^{(\epsilon)}_\gamma$ for general paths γ by demanding that this relation is satisfied exactly (rather than merely up to elements from $\ker(\Delta)$) and that $\psi^{(\epsilon)}_e = \psi_e$ for any edge e. Freeness of the groupoid $\pi_1(X_1; X_0)$ guarantees that this definition is well-posed. More explicitly, for a path $\gamma = e_n e_{n-1} \dots e_1$ it gives

$$\psi^{(\epsilon)}_\gamma = \psi_{e_n}\,\left(\epsilon_{e_n} \rhd \psi_{e_{n-1}}\right) \dots \left(\epsilon_{e_n} \dots \epsilon_{e_2} \rhd \psi_{e_1}\right). \tag{2.23}$$

By induction on n, formulas (2.18) and (2.22) imply that with this definition of $\psi^{(\epsilon)}_\gamma$, transformation law (2.19) is indeed satisfied for any γ. We also note that the composition rule (2.22) yields the inversion formula

$$\psi^{(\epsilon)}_{\gamma^{-1}} = \left(\epsilon_\gamma^{-1} \rhd \psi^{(\epsilon)}_\gamma\right)^{-1}. \tag{2.24}$$

Let us now return to the problem of defining an action of edge transformations on $\boldsymbol{\varphi}$. The guiding principle is the preservation of the fake flatness condition. Thus we must have

$$\Delta\varphi'_f = \epsilon'_{\partial f} = \Delta\psi^{(\epsilon)}_{\partial f}\,\epsilon_{\partial f} = \Delta\left(\psi^{(\epsilon)}_{\partial f}\,\varphi_f\right). \tag{2.25}$$

The simplest way to satisfy this condition is to declare

$$\varphi'_f = \psi^{(\epsilon)}_{\partial f}\,\varphi_f. \tag{2.26}$$

We illustrate the above definitions by considering a field configuration on the geometry depictured on figure 12. Such configuration consists of five elements $\epsilon_{e_i} \in \mathcal{E}$ and two

$\varphi_{f_i} \in \Phi$. Vertex transformation given by the collection $\{\psi_{e_i}\}$ maps ϵ_{e_i} to $\Delta\psi_{e_i}\,\epsilon_{e_i}$. Action on φ variables is given by

$$\varphi_{f_1} \mapsto \psi_{e_4}\,\left(\epsilon_{e_4} \triangleright \psi_{e_5}\right)\,\left(\epsilon_{e_4}\epsilon_{e_5} \triangleright \psi_{e_1}\right)\,\varphi_{f_1}, \tag{2.27a}$$

$$\varphi_{f_2} \mapsto \underbrace{\left(\epsilon_{e_5}^{-1} \triangleright \psi_{e_5}^{-1}\right)}_{\psi_{e_5^{-1}}}\,\left(\epsilon_{e_5}^{-1} \triangleright \psi_{e_3}\right)\,\left(\epsilon_{e_5}^{-1}\epsilon_{e_3} \triangleright \psi_{e_2}\right)\,\varphi_{f_2}. \tag{2.27b}$$

Definition (2.26) implies the following transformation law for φ_σ for arbitrary σ:

$$\varphi'_\sigma = \psi^{(\epsilon)}_{\partial\sigma}\,\varphi_\sigma. \tag{2.28}$$

This can be proven as follows. First we note that, by definition, it holds for $\sigma = f$ for any face f. Secondly, if σ and σ' share the base point and are such that (2.28) holds, then the same is true for the product $\sigma\sigma'$:

$$\begin{aligned}
\varphi_\sigma\,\varphi_{\sigma'} \longmapsto \psi^{(\epsilon)}_{\partial\sigma}\,\varphi_\sigma\,\psi^{(\epsilon)}_{\partial\sigma'}\,\varphi_{\sigma'} &= \psi^{(\epsilon)}_{\partial(\sigma\sigma')}\,\left(\epsilon_{\partial\sigma} \triangleright \psi^{(\epsilon)}_{\partial\sigma'}\right)^{-1}\,\varphi_\sigma\,\psi^{(\epsilon)}_{\partial\sigma'}\,\varphi_{\sigma'}\\
&= \psi^{(\epsilon)}_{\partial(\sigma\sigma')}\,\left(\epsilon_{\partial\sigma} \triangleright \psi^{(\epsilon)}_{\partial\sigma'}\right)^{-1}\,\varphi_\sigma\,\psi^{(\epsilon)}_{\partial\sigma'}\,\varphi_\sigma^{-1}\,\varphi_{\sigma\sigma'}\\
&= \psi^{(\epsilon)}_{\partial(\sigma\sigma')}\,\left(\epsilon_{\partial\sigma} \triangleright \psi^{(\epsilon)}_{\partial\sigma'}\right)^{-1}\,\left(\Delta\varphi_\sigma \triangleright \psi^{(\epsilon)}_{\partial\sigma'}\right)\,\varphi_{\sigma\sigma'}\\
&\overset{\text{f.f.}}{=} \psi^{(\epsilon)}_{\partial(\sigma\sigma')}\,\left(\epsilon_{\partial\sigma} \triangleright \psi^{(\epsilon)}_{\partial\sigma'}\right)^{-1}\,\left(\epsilon_{\partial\sigma} \triangleright \psi^{(\epsilon)}_{\partial\sigma'}\right)\,\varphi_{\sigma\sigma'} = \psi^{(\epsilon)}_{\partial(\sigma\sigma')}\,\varphi_{\sigma\sigma'}.
\end{aligned} \tag{2.29}$$

Next we show that if σ is such that (2.28) holds and $b(\sigma) = s(\gamma)$, then (2.28) holds also for $\gamma \triangleright \sigma$. Indeed, in this situation we have

$$\varphi_\sigma \longmapsto \psi^{(\epsilon)}_{\partial\sigma}\,\varphi_\sigma, \qquad \epsilon_\gamma \longmapsto \Delta\psi^{(\epsilon)}_\gamma\,\epsilon_\gamma. \tag{2.30}$$

Since $\partial(\gamma \triangleright \sigma) = \gamma\,\partial\sigma\,\gamma^{-1}$, we have to check that

$$\epsilon_\gamma \triangleright \varphi_\sigma \longmapsto \psi^{(\epsilon)}_{\gamma\,\partial\sigma\,\gamma^{-1}}\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right). \tag{2.31}$$

This is indeed the case, since $\epsilon_\gamma \triangleright \varphi_\sigma$ transforms as:

$$\begin{aligned}
\epsilon_\gamma \triangleright \varphi_\sigma \longmapsto \left(\Delta\psi^{(\epsilon)}_\gamma\,\epsilon_\gamma\right) \triangleright \left(\psi_{\partial\sigma}\,\varphi_\sigma\right) &= \psi^{(\epsilon)}_\gamma\,\left(\epsilon_\gamma \triangleright \left(\psi^{(\epsilon)}_{\partial\sigma}\,\varphi_\sigma\right)\right)\,\psi^{(\epsilon)\,-1}_\gamma\\
&= \psi^{(\epsilon)}_\gamma\,\left(\epsilon_\gamma \triangleright \psi^{(\epsilon)}_{\partial\sigma}\right)\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)\,\psi^{(\epsilon)\,-1}_\gamma\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)^{-1}\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)\\
&= \psi^{(\epsilon)}_{\gamma\,\partial\sigma}\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)\,\psi^{(\epsilon)\,-1}_\gamma\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)^{-1}\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)\\
&= \psi^{(\epsilon)}_{\gamma\,\partial\sigma}\,\left(\Delta\left(\epsilon_\gamma \triangleright \varphi_\sigma\right) \triangleright \psi^{(\epsilon)\,-1}_\gamma\right)\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)\\
&= \psi^{(\epsilon)}_{\gamma\,\partial\sigma}\,\left(\left(\epsilon_\gamma\,\Delta\varphi_\sigma\,\epsilon_\gamma^{-1}\right) \triangleright \psi^{(\epsilon)\,-1}_\gamma\right)\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right)\\
&= \psi^{(\epsilon)}_{\gamma\,\partial\sigma}\,\left(\epsilon_{\gamma\,\partial\sigma\,\gamma^{-1}} \triangleright \psi^{(\epsilon)\,-1}_\gamma\right)\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right) = \psi^{(\epsilon)}_{\gamma\,\partial\sigma\,\gamma^{-1}}\,\left(\epsilon_\gamma \triangleright \varphi_\sigma\right).
\end{aligned} \tag{2.32}$$

This concludes the proof, since any σ may be written as a product of some number of elements of the form $\gamma \triangleright f$.

The following special case of the above result is worth to be mentioned separately: if σ has trivial boundary ($\partial\sigma = 1$), then φ_σ is invariant with respect to edge transformations.

$$\begin{tikzcd}
(\boldsymbol{\epsilon},\boldsymbol{\varphi}) \arrow[d, "\{\psi_e\}"'] \arrow[dr, "\{\psi'_e\,\psi_e\}"] & \\
(\boldsymbol{\epsilon}',\boldsymbol{\varphi}') \arrow[r, "\{\psi'_e\}"] & (\boldsymbol{\epsilon}'',\boldsymbol{\varphi}'')
\end{tikzcd}$$

Figure 14. Edge transformation $\{\psi_e\}$ followed by $\{\psi'_e\}$ coincides with $\{\psi'_e\,\psi_e\}$.

$$\begin{tikzcd}
(\boldsymbol{\epsilon},\boldsymbol{\varphi}) \arrow[r, "\{\xi_v\}"] \arrow[d, "\{\psi_e\}"'] & (\boldsymbol{\epsilon}',\boldsymbol{\varphi}') \arrow[d, "\{\xi_{t(e)}\triangleright\psi_e\}"] \\
(\boldsymbol{\epsilon}'',\boldsymbol{\varphi}'') \arrow[r, "\{\xi_v\}"] & (\boldsymbol{\epsilon}''',\boldsymbol{\varphi}''')
\end{tikzcd}$$

Figure 15. Illustration of the semi-direct product structure of the group $\mathcal{E}_X^{(0)} \ltimes \Phi_X^{(1)}$ generated by vertex and edge transformations.

Edge transformations form a group $\Phi_X^{(1)}$, with composition computed edge-wise (see figure 14): transformation $\{\psi_e\}$ followed by $\{\psi'_e\}$ coincides with $\{\psi''_e\}$, where $\psi''_e = \psi'_e\,\psi_e$. Indeed, for configurations as on figure 14 we have

$$\epsilon''_e = \Delta\psi'_e\,\epsilon'_e = \Delta\psi'_e\,\Delta\psi_e\,\epsilon_e = \Delta\psi''_e\,\epsilon_e, \tag{2.33a}$$

$$\varphi''_f = \psi'^{(\epsilon')}_{\partial f}\,\varphi'_f = \psi'^{(\epsilon')}_{\partial f}\,\psi^{(\epsilon)}_{\partial f}\,\varphi_f. \tag{2.33b}$$

Thus in order to prove the claimed composition law it only remains to show that $\psi''^{(\epsilon)}_{\partial f} = \psi'^{(\epsilon')}_{\partial f}\,\psi^{(\epsilon)}_{\partial f}$. In fact even more is true: for any path γ we have

$$\psi''^{(\epsilon)}_{\gamma} = \psi'^{(\epsilon')}_{\gamma}\,\psi^{(\epsilon)}_{\gamma} \tag{2.34}$$

Indeed, by the induction principle, it is sufficient to demonstrate that the above equality is satisfied for a composite path $\gamma\gamma'$ provided that it holds for γ and γ' separately. To this end we use (2.22) and apply the inductive hypothesis:

$$\begin{aligned}
\psi''^{(\epsilon)}_{\gamma\gamma'} &= \psi''^{(\epsilon)}_{\gamma}\left(\epsilon_\gamma \triangleright \psi''^{(\epsilon)}_{\gamma'}\right) = \psi'^{(\epsilon')}_{\gamma}\,\psi^{(\epsilon)}_{\gamma}\left(\epsilon_\gamma \triangleright \left(\psi'^{(\epsilon')}_{\gamma'}\,\psi^{(\epsilon)}_{\gamma'}\right)\right) \\
&= \psi'^{(\epsilon')}_{\gamma}\,\underbrace{\psi^{(\epsilon)}_{\gamma}\left(\epsilon_\gamma \triangleright \psi'^{(\epsilon')}_{\gamma'}\right)\psi^{(\epsilon)-1}_{\gamma}}_{\text{Peiffer}}\,\underbrace{\psi^{(\epsilon)}_{\gamma}\left(\epsilon_\gamma \triangleright \psi^{(\epsilon)}_{\gamma'}\right)}_{\psi^{(\epsilon)}_{\gamma\gamma'}} \\
&= \psi'^{(\epsilon')}_{\gamma}\left(\left(\Delta\psi^{(\epsilon)}_{\gamma}\,\epsilon_\gamma\right) \triangleright \psi'^{(\epsilon')}_{\gamma'}\right)\psi^{(\epsilon)}_{\gamma\gamma'} = \psi'^{(\epsilon')}_{\gamma\gamma'}\,\psi^{(\epsilon)}_{\gamma\gamma'}.
\end{aligned} \tag{2.35}$$

We remark also that conjugation of an edge transformation with a vertex transformation gives another edge transformation, see figure 15. This means that vertex transformations together with edge transformations form a semi-direct product structure $\mathcal{E}_X^{(0)} \ltimes \Phi_X^{(1)}$.

Next we define **plaquette transformations**. They are labeled by $\ker(\Delta)$-valued collections $\boldsymbol{\chi} = \{\chi_f\}$ indexed by faces. The action on fields is given by

$$\epsilon'_e = \epsilon_e, \qquad \varphi'_f = \chi_f \, \varphi_f. \tag{2.36}$$

It is clear that the fake flatness condition is preserved.

As announced at the beginning of this subsection, in topological field theories based on crossed modules all edge transformations are regarded as gauge transformations. We will now list some important consequences of this choice:

1. Up to a gauge transformation, $\boldsymbol{\epsilon}$ is uniquely determined by $\overline{\boldsymbol{\epsilon}}$. Thus the only gauge invariant functions constructed entirely of $\boldsymbol{\epsilon}$ are the (conjugacy classes of) holonomies of $\overline{\boldsymbol{\epsilon}}$, which are topological observables.

2. Apart from topological degrees of freedom present in $\overline{\boldsymbol{\epsilon}}$, there remain $\ker(\Delta)$-valued degrees of freedom in $\boldsymbol{\varphi}$. These can be made topological by introducing additional flatness constraint: $\varphi_{\partial q} = 1$ for every ball q.

3. There exists a space $B\mathbb{G}$, called the **classifying space** of $\mathbb{G}$, with the property that gauge equivalence classes of flat field configurations are in one-to-one correspondence with homotopy classes of maps from X to $B\mathbb{G}$. In particular the set of gauge orbits of flat gauge fields is a homotopy invariant of X. We review this in appendix C.

4. Despite the fact that models under consideration are formulated in terms of the crossed module $\mathbb{G}$, they depend only on its weak equivalence class. We will obtain this fact as a corollary from considerations in section 3.4. Furthermore, we give its second, logically independent proof in appendix C.4.

Here we would like to focus on an alternative possibility and regard only edge transformations with $\psi_e \in \ker(\Delta)$ for each edge e as gauge redundancies. With this definition it is possible to formulate dynamical models with 1-form and 2-form gauge fields interacting in an interesting way. Indeed, the conjugacy classes of holonomies of $\boldsymbol{\epsilon}$ (rather than merely their reductions modulo $\mathrm{im}(\Delta)$) become gauge invariant. These holonomies are not necessarily trivial for contractible loops, so some non-topological degrees of freedom may be present in the field $\boldsymbol{\epsilon}$.

Topological quantum field theories briefly discussed above may still be recovered in a certain limit, by enforcing invariance with respect to all edge transformations and flatness of the $\boldsymbol{\varphi}$ field dynamically. Furthermore, two other well-known models may be obtained as special cases:

- If $\mathcal{E}$ is taken to be trivial, Φ can still be any abelian group. In this case one recovers 2-form lattice gauge theory valued in Φ.

- Taking $\Phi = \mathcal{E}$, homomorphism Δ to be the identity map and the action of $\mathcal{E}$ on Φ given by conjugation we recover the standard lattice gauge theory.

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There are two other special cases which correspond to slight variations of the above:

- Given any $\mathcal{E}$ and an abelian group Φ on which $\mathcal{E}$ acts one can form a crossed module by letting Δ be the trivial homomorphism. Then fake flatness implies that $\boldsymbol{\epsilon}$ is flat, so it carries no local gauge-invariant degrees of freedom. The effect of nontrivial holonomies of $\boldsymbol{\epsilon}$ along non-contractible loops may be loosely described as imposing twisted boundary conditions for the field $\boldsymbol{\varphi}$. Models of this type may be obtained from 2-form gauge theories by gauging a global symmetry of the form $\varphi_f \mapsto \epsilon \rhd \varphi_f$.

- Taking a crossed module with injective Δ one obtains lattice gauge theory with gauge group $\mathcal{E}$ in which the curvature is constrained to be valued in the normal subgroup $\mathrm{im}(\Delta) \cong \Phi$.

We have shown that proposed models unify and at the same time generalize several interesting classes of gauge theories involving 1-form and 2-form gauge fields, which provides compelling motivation to study them.

We close this subsection with a technical lemma, to be used later. Consider the effect of a vertex transformation $\boldsymbol{\xi}$ valued in $\mathrm{im}(\Delta)$ on a configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$. For each vertex v we write $\xi_v = \Delta(\rho_v)$ for some $\rho_v \in \Phi$. Then we have

$$\begin{aligned} \epsilon'_e &= \Delta(\rho_{t(e)})\, \epsilon_e\, \Delta(\rho_{s(e)}^{-1}) \;=\; \Delta(\rho_{t(e)})\, \overbrace{\epsilon_e\, \Delta(\rho_{s(e)}^{-1})\, \epsilon_e^{-1}}^{\text{Peiffer}}\, \epsilon_e \\ &= \Delta(\rho_{t(e)})\, \Delta(\epsilon_e \rhd \rho_{s(e)}^{-1})\, \epsilon_e \;=\; \Delta(\psi_e)\, \epsilon_e, \end{aligned} \tag{2.37}$$

where $\psi_e = \rho_{t(e)} \left(\epsilon_e \rhd \rho_{s(e)}^{-1}\right)$. We claim that one also has $\varphi'_f = \psi^{(\epsilon)}_{\partial_f} \varphi_f$, so the pertinent vertex transformation is equivalent to an edge transformation with some $\boldsymbol{\psi}$ depending on $\boldsymbol{\rho}$ and $\boldsymbol{\epsilon}$. Indeed, first observe that we have

$$\begin{aligned} \varphi'_f &= \xi_{b(f)} \rhd \varphi_f = \Delta\rho_{b(f)} \rhd \varphi_f \overset{\text{Peiffer}}{=} \rho_{b(f)}\, \overbrace{\varphi_f\, \rho_{b(f)}^{-1}\, \varphi_f^{-1}}^{\text{Peiffer}}\, \varphi_f \\ &= \rho_{b(f)} \left(\Delta\varphi_f \rhd \rho_{b(f)}^{-1}\right) \varphi_f \overset{\text{f.f.}}{=} \rho_{b(f)} \left(\epsilon_{\partial_f} \rhd \rho_{b(f)}^{-1}\right) \varphi_f. \end{aligned} \tag{2.38}$$

It only remains to show that $\psi^{(\epsilon)}_{\partial f} = \rho_{b(f)} \left(\epsilon_{\partial f} \rhd \rho_{b(f)}^{-1}\right)$. In fact even more is true: $\psi^{(\epsilon)}_{\gamma} = \rho_{t(\gamma)} \left(\epsilon_\gamma \rhd \rho_{s(\gamma)}^{-1}\right)$ for any path γ. This follows from (2.22) and the definition of $\boldsymbol{\psi}$, by induction on the number of edges in γ.

2.4 Interesting field configurations — examples

In this subsection we present examples of field configurations illustrating certain phenomena that will play important roles in the further discussion.

Firstly, we would like to point out that flatness of $\overline{\boldsymbol{\epsilon}}$ does not guarantee that one can find a corresponding *flat* $\boldsymbol{\epsilon}$. To illustrate this feature we consider the decomposition of a 2-torus depictured on figure 16.

We take $\mathcal{E}$ to be the dihedral group D_4. It is generated by elements x, y, z, which are subject to relations

$$x^2 = y^2 = z^2 = 1, \quad xz = zx, \quad yz = zy, \quad xy = zyx. \tag{2.39}$$

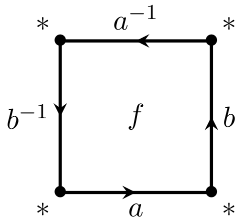

Figure 16. A decomposition of the 2-torus. In this case we have $\partial f = b^{-1}a^{-1}ba$.

The group Φ is taken to be $\{1, z\}$, with Δ the inclusion map. Action of $\mathcal{E}$ on Φ is trivial. As a result, $\ker(\Delta)$ is trivial and $\operatorname{coker}(\Delta) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now consider a configuration with $\overline{\epsilon_a} = \overline{x}$ and $\overline{\epsilon_b} = \overline{y}$. Since $\partial f = b^{-1}a^{-1}ba$ and $\operatorname{coker}(\Delta)$ is abelian, we have $\overline{\epsilon_{\partial f}} = 1$. On the other hand we must have $\epsilon_a \in \{x, zx\}$. Similarly, $\epsilon_b \in \{y, zy\}$. In each of the four possibilities we get $\epsilon_{\partial f} = z$. Therefore $\boldsymbol{\epsilon}$ cannot be flat for the given $\overline{\boldsymbol{\epsilon}}$.

Next we would like to show that there may exist field configurations which are not gauge equivalent, even though all holonomies coincide.[8] To this end we continue to consider the 2-torus, but we no longer restrict ourselves to the specific choice of the crossed module.

We choose $\epsilon_a = \epsilon_b = 1$. Notice that this condition is preserved by vertex and edge transformations with arbitrary ξ_* and ψ_a, ψ_b. Furthermore, φ_f may be any element from the kernel of Δ. Under a gauge transformation, this element changes according to the formula

$$\varphi_f \mapsto \xi_* \rhd (\psi_{b^{-1}a^{-1}ba}\, \varphi_f) \,. \tag{2.40}$$

Next, since $\ker(\Delta)$ is abelian,

$$\psi_{b^{-1}a^{-1}ba} \overset{\epsilon=1}{=} \psi_{b^{-1}}\, \psi_{a^{-1}}\, \psi_b\, \psi_a \overset{\epsilon=1}{=} \psi_b^{-1}\, \psi_a^{-1}\, \psi_b\, \psi_a = 1. \tag{2.41}$$

Therefore the formula (2.40) simplifies to

$$\varphi_f \mapsto \xi_* \rhd \varphi_f, \tag{2.42}$$

which does not depend on the choice of ψ_a and ψ_b in $\ker(\Delta)$. This means that configurations with $\boldsymbol{\epsilon} = \mathbf{1}$ and φ_f in different orbits of $\mathcal{E}$ are not related by a gauge transformation. Thus there will be at least two such non-equivalent configurations if $\ker(\Delta)$ is nontrivial. On the other hand, all these configurations have the same values of all holonomies. Indeed, 1-holonomies are all equal 1 and there are no nontrivial 2-holonomies, since the second homotopy group of a torus vanishes. This indicates existence of gauge invariant observables associated to non-spherical surfaces. This is indeed true, but they are slightly tricky to define. We will not consider this problem here. An interesting discussion in the context of state sum formulation of topological higher gauge theories was given in [29].

[8]This is true regardless of the choice whether edge transformations not valued in $\ker(\Delta)$ are regarded as gauge transformations.

In the final example of this subsection we shall show that some 2-holonomies may be determined already by $\overline{\boldsymbol{\epsilon}}$. In particular, it may happen that for some $\overline{\boldsymbol{\epsilon}}$ it is not possible to choose $\boldsymbol{\epsilon}$ and $\boldsymbol{\varphi}$ so that $\varphi_{\partial q} = 1$ for every ball q.

Let us consider the decomposition of the projective plane presented on figure 8. We take the crossed module with

$$\mathcal{E} = \Phi = \mathbb{Z}_4, \quad \Delta(n) = 2n, \quad m \triangleright n = (-1)^m n, \tag{2.43}$$

where we use additive notation. In this case $\ker(\Delta) \cong \operatorname{coker}(\Delta) \cong \mathbb{Z}_2$.

In the present example, fake flatness does not impose any conditions on $\overline{\epsilon_e}$. Thus we can set it to be the nonzero element of $\mathbb{Z}_2$. Then $\epsilon_e \in \{1, 3\}$, so $\epsilon_{\partial f} = \epsilon_{e^2} = 2$. Then fake flatness gives $\Delta(\varphi_f) = 2$, so $\varphi_f \in \{1, 3\}$.

Recall now that the second homotopy group of $\mathbb{RP}^2$ is generated by the element $\sigma = (e \triangleright f)\, f^{-1}$. Evaluation of the 2-holonomy along this generator gives

$$\varphi_\sigma = (\epsilon_e \triangleright \varphi_f) - \varphi_f = 2\varphi_f = 2, \tag{2.44}$$

regardless of which of the two possible values of ϵ_e and φ_f have been chosen. Similarly one can show that if $\overline{\epsilon_e}$ is trivial, then $\varphi_\sigma = 0$.

One can also embed $\mathbb{RP}^2$ in $\mathbb{RP}^3$ by attaching an additional 3-cell q along the generator of $\pi_2(\mathbb{RP}^2)$. In other words, we may decompose $\mathbb{RP}^3$ into $\mathbb{RP}^2$ and an extra ball q with $\partial q = \sigma$. Field configurations discussed above make sense also as configurations on $\mathbb{RP}^3$. Hence we see that in this case if $\overline{\boldsymbol{\epsilon}}$ is nontrivial, then $\boldsymbol{\varphi}$ cannot be chosen to be flat. As reviewed in the appendix C.3, this phenomenon is controlled by the so-called **Postnikov class**.

3 Hamiltonian models

3.1 Construction

In this section we present the proper construction of our models. We work in the hamiltonian formulation of quantum mechanics. As the first step we construct the Hilbert space $\mathcal{H}$ and local operators resembling electric field operators from the usual gauge theory. Hamiltonian H_{E} is defined in terms of these electric operators. It may be though of as a kinetic term. The full hamiltonian H involves also a magnetic term H_{M}. Each of H_{E} and H_{M} is separately solvable (being a sum of commuting local terms), but its action exchanges states which are stationary for the other. Thus the sum is expected to describe interesting dynamics.

Let us consider the Hilbert space $\mathcal{H}_0$ with an orthonormal basis whose elements are labeled by collections $\boldsymbol{\epsilon} = \{\epsilon_e\}$, $\boldsymbol{\varphi} = \{\varphi_f\}$ of elements of $\mathcal{E}$ and Φ,

$$\mathcal{H}_0 \cong \Big(\bigotimes_e L^2(\mathcal{E}) \Big) \otimes \Big(\bigotimes_f L^2(\Phi) \Big). \tag{3.1}$$

The Hilbert space of the constructed model will be the subspace $\mathcal{H} \subset \mathcal{H}_0$ spanned by those $|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle$ for which the fake flatness condition $\Delta(\varphi_f) = \epsilon_{\partial f}$ is satisfied. This Hilbert space is not the tensor product of local Hilbert spaces associated to edges and faces, but it does admit a basis consisting of product states.

Several interesting classes of operators may be defined on $\mathcal{H}$:

- For a collection $\boldsymbol{\xi} = \{\xi_v\}$ of elements of $\mathcal{E}$ we define $\mathsf{G}(\boldsymbol{\xi})$ by

$$\mathsf{G}(\boldsymbol{\xi})\,|\boldsymbol{\epsilon},\boldsymbol{\varphi}\rangle = \left|\{\xi_{t(e)}\,\epsilon_e\,\xi_{s(e)}^{-1}\},\{\xi_{b(f)} \triangleright \varphi_f\}\right\rangle. \tag{3.2}$$

- For a collection $\boldsymbol{\psi} = \{\psi_e\}$ of elements of Φ we let $\mathsf{V}(\boldsymbol{\psi})$ be

$$\mathsf{V}(\boldsymbol{\psi})\,|\boldsymbol{\epsilon},\boldsymbol{\varphi}\rangle = \left|\{\Delta(\psi_e)\,\epsilon_e\},\{\psi^{(\epsilon)}_{\partial f}\,\varphi_f\}\right\rangle. \tag{3.3}$$

- For a collection $\boldsymbol{\chi} = \{\chi_f\}$ of elements of $\ker(\Delta)$ we introduce $\mathsf{W}(\boldsymbol{\chi})$ by putting

$$\mathsf{W}(\boldsymbol{\chi})\,|\boldsymbol{\epsilon},\boldsymbol{\varphi}\rangle = |\boldsymbol{\epsilon},\{\chi_f\,\varphi_f\}\rangle\,. \tag{3.4}$$

Operators $\mathsf{G}(\boldsymbol{\xi})$ form a representation of the group of vertex transformations $\mathcal{E}^{(0)}_X$ on $\mathcal{H}$. We will call them vertex Gauss' operators. Only elements of $\mathcal{H}$ which satisfy the vertex Gauss' law, i.e. are invariant with respect to the action of all $\mathsf{G}(\boldsymbol{\xi})$, will be regarded as physical states.

Operators $\mathsf{V}(\boldsymbol{\psi})$ form a representation of the group $\Phi^{(1)}_X$ of edge transformations. As discussed in subsection 2.3, its subgroup $\ker(\Delta)^{(1)}_X$ describes (a part of the) gauge redundancy of the constructed model. Therefore we will call $\mathsf{V}(\boldsymbol{\psi})$ with $\psi_e \in \ker(\Delta)$ the edge Gauss' operators. The final requirement for an element of $\mathcal{H}$ to be regarded as a physical state is that it should satisfy the edge Gauss' law, i.e. be invariant with respect to the action of all edge Gauss' operators.

We will now construct electric operators associated to edges. These are required to be gauge invariant, i.e. to commute with Gauss' operators of both types. Let us denote by $\mathsf{V}_e(\psi)$ the operator $\mathsf{V}(\{\psi_{e'}\})$ with $\psi_{e'} = \psi$ for $e' = e$ and $\psi_{e'} = 1$ otherwise. Recall that $\ker(\Delta)$ is a central subgroup of Φ, so operators $\mathsf{V}_e(\psi)$ do commute with all edge Gauss' operators. However, they are not invariant with respect to vertex gauge transformations. Instead we have

$$\mathsf{G}(\boldsymbol{\xi})\mathsf{V}_e(\psi)\mathsf{G}(\boldsymbol{\xi})^{-1} = \mathsf{V}_e(\xi_{t(e)} \triangleright \psi). \tag{3.5}$$

This means that to obtain a gauge invariant operator it is sufficient to sum $V_e(\psi)$ over ψ with any $\mathcal{E}$-invariant weight $\mu : \Phi \to \mathbb{C}$. Explicitly, we define

$$\mathsf{V}_{e,\mu} = \sum_{\psi\in\Phi} \mu(\psi)\mathsf{V}_e(\psi). \tag{3.6}$$

This operator commutes with $\mathsf{G}(\boldsymbol{\xi})$ provided that $\mu(\xi \triangleright \psi) = \mu(\psi)$ for all $\xi \in \mathcal{E}$.

If μ and μ' are two $\mathcal{E}$-invariant functions, operators $\mathsf{V}_{e,\mu}$ and $\mathsf{V}_{e',\mu'}$ commute. This is obvious for $e' \neq e$, while:

$$\begin{aligned}\mathsf{V}_{e,\mu}\mathsf{V}_{e,\mu'} &= \sum_{\psi,\psi'\in\Phi} \mu(\psi)\mu'(\psi')\mathsf{V}_e(\psi\psi') = \sum_{\psi,\psi'\in\Phi} \mu(\psi)\mu'(\psi')\mathsf{V}_e(\underbrace{\psi\psi'\psi^{-1}}_{\psi''})\mathsf{V}_e(\psi)\\ &= \sum_{\psi,\psi''\in\Phi} \mu(\psi)\mu'(\psi^{-1}\psi''\psi)\mathsf{V}_e(\psi'')\mathsf{V}_e(\psi) = \mathsf{V}_{e,\mu'}\mathsf{V}_{e,\mu}.\end{aligned} \tag{3.7}$$

Operators $\mathsf{W}(\boldsymbol{\chi})$ form a representation of the abelian group $\ker(\Delta)^{(2)}_X$ and, in particular, commute with each other. Moreover, they commute with all $\mathsf{V}(\boldsymbol{\psi})$. We will use them to construct electric operators associated to faces.

Let us denote by $\mathsf{W}_f(\chi)$ the operator $\mathsf{W}(\{\chi'_f\})$ with $\chi'_f = \chi$ for $f' = f$ and $\chi'_f = 1$ for $f' \neq f$. For a function $\nu : \ker(\Delta) \to \mathbb{C}$ we put

$$\mathsf{W}_{f,\nu} = \sum_{\chi \in \ker(\Delta)} \nu(\chi) \mathsf{W}_f(\chi). \tag{3.8}$$

The gauge transformation law for $\mathsf{W}_f(\chi)$ operators takes the form

$$\mathsf{G}(\boldsymbol{\xi})\mathsf{W}_f(\chi)\mathsf{G}(\boldsymbol{\xi})^{-1} = \mathsf{W}_f(\xi_{b(f)} \triangleright \chi), \tag{3.9}$$

hence $\mathsf{W}_{f,\nu}$ commutes with $\mathsf{G}(\boldsymbol{\xi})$ if and only if $\nu(\xi \triangleright \chi) = \nu(\chi)$ for all $\xi \in \mathcal{E}$.

As our candidate for the electric hamiltonian we take

$$\mathsf{H}_{\mathrm{E}} = \mathsf{H}_V + \mathsf{H}_W, \qquad \text{where} \qquad \mathsf{H}_V = \sum_e \mathsf{V}_{e,\mu_e}, \qquad \mathsf{H}_W = \sum_f \mathsf{W}_{f,\nu_f} \tag{3.10}$$

with a priori different functions μ_e and ν_f for different edges e and faces f, since the spatial lattice is not necessarily assumed to admit any symmetries. By construction, H_{E} commutes with all Gauss' operators and thus is a well-defined operator on the physical subspace of $\mathcal{H}$. In order for H_{E} to be self-adjoint we have to take functions μ, ν to satisfy $\mu(\psi^{-1}) = \overline{\mu(\psi)}$ and $\nu(\chi^{-1}) = \overline{\nu(\chi)}$. Furthermore, we would like H_{E} to admit either a unique ground state, or at most a finite number of ground states, dependent only on the topology. This can be achieved by assuming that all functions μ_e and ν_f are such that their Fourier transforms vanish at the trivial representation and are positive otherwise,[9] as will be demonstrated in subsection 3.4.

Following a common terminology we shall call operators, which are diagonal in the adapted basis of $\mathcal{H}$, "magnetic". The first important class of operators of this type are those constructed out of 1-holonomies. Consider a function $\eta : \mathcal{E} \to \mathbb{C}$ and a path γ. We define an operator $\mathsf{A}_{\gamma,\eta}$ by

$$\mathsf{A}_{\gamma,\eta}|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = \eta(\epsilon_\gamma)|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle. \tag{3.11}$$

This operator is gauge invariant if and only if the endpoints of γ coincide and η is a class function, i.e. $\eta(\xi\,\epsilon\,\xi^{-1}) = \eta(\epsilon)$ for any $\xi, \epsilon \in \mathcal{E}$. Our magnetic hamiltonian will involve only terms $\mathsf{A}_{\partial f,\eta}$ for faces f, as in standard lattice gauge theory. In this case the function η needs to be defined only on the subgroup $\mathrm{im}(\Delta) \subseteq \mathcal{E}$, since $\epsilon_{\partial f} \in \mathrm{im}(\Delta)$ by fake flatness.

Analogously, let θ be a complex function on Φ and let $\sigma \in \pi_2(X_2, X_1; x)$ for some base point $x \in X_0$. We define an operator $\mathsf{B}_{\sigma,\theta}$ by

$$\mathsf{B}_{\sigma,\theta}|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = \theta(\varphi_\sigma)|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle. \tag{3.12}$$

Recall that for σ with trivial boundary the 2-holonomy along σ is invariant with respect to edge gauge transformations. Thus $\mathsf{B}_{\sigma,\theta}$ commutes with each $\mathsf{V}(\boldsymbol{\psi})$.

[9] It can be shown that functions satisfying this condition as well as the required invariance properties always exist.

We note also that if $\partial\sigma = 1$, function θ needs to be defined only on $\ker(\Delta)$, since then $\Delta(\varphi_\sigma) = 1$ by fake flatness.

Recall that 2-holonomies transform nontrivially under vertex transformation. This has the consequence that $\mathsf{B}_{\sigma,\eta}$ commutes with all $\mathsf{G}(\boldsymbol{\xi})$ (and thus defines an operator on the physical subspace of $\mathcal{H}$) provided that η satisfies

$$\eta(\xi \triangleright \varphi) = \eta(\varphi) \quad \text{for all} \quad \varphi \in \ker(\Delta),\ \xi \in \mathcal{E}. \tag{3.13}$$

We are ready to propose our candidate for the magnetic hamiltonian:

$$\mathsf{H}_{\mathrm{M}} = \mathsf{H}_A + \mathsf{H}_B, \qquad \text{where} \qquad \mathsf{H}_A = \sum_f \mathsf{A}_{\partial f,\eta_f}, \qquad \mathsf{H}_B = \sum_q \mathsf{B}_{\partial q,\theta_q} \tag{3.14}$$

with a priori different functions η_f and θ_q for different faces f and balls q. We shall assume that functions η_f and θ_q are non-negative, with value zero attained only for the neutral element. Thus the magnetic hamiltonian penalizes configurations with nontrivial holonomies along contractible loops and surfaces.

We close this subsection with a brief discussion on how the above construction needs to be modified if all edge transformations (i.e. with ψ_e not necessarily in $\ker(\Delta)$) are regarded as gauge transformations. In this case the term in H_{M} involving A operators has to be dropped, as it is no longer gauge invariant. Furthermore, all V operators in H_{E} may be dropped, since they act trivially on the space of physical states. Thus the hamiltonian reduces to $\mathsf{H}_B + \mathsf{H}_W$.

3.2 An explicit example

In order to illustrate general features discussed so far, we consider here an example constructed on a cubic lattice with a particular crossed module chosen. We shall parametrize the set of vertices by ordered triples of integers $[j_1, j_2, j_3]$, edges by ordered triples consisting of two integers and one half-integer, while for faces we use ordered triples consisting of an integer and two half-integers - see figure 17.

Due to the translational symmetry we can restrict attention to one elementary cell. We will introduce the necessary notation and conventions based upon this cell, in order to avoid tedious expressions.

The orientation of edges is chosen as follows:

$$s(e_{[\frac{1}{2},0,0]}) = v_{[0,0,0]}, \qquad t(e_{[\frac{1}{2},0,0]}) = v_{[1,0,0]}, \tag{3.15}$$

and similarly for other edges, while faces are oriented so that:

$$\partial f_{[\frac{1}{2},\frac{1}{2},0]} = e^{-1}_{[0,\frac{1}{2},0]}\, e^{-1}_{[\frac{1}{2},1,0]}\, e_{[1,\frac{1}{2},0]}\, e_{[\frac{1}{2},0,0]}, \tag{3.16}$$

and analogously for other faces. We illustrate this on the figure 17. The basepoints are chosen so that $b(f_{[\frac{1}{2},\frac{1}{2},0]}) = b(f_{[\frac{1}{2},0,\frac{1}{2}]}) = b(f_{[0,\frac{1}{2},\frac{1}{2}]}) = v_{[0,0,0]}$, and so on.

Finally, each 3-cell q will be parameterized by an ordered triple of half-integers and oriented so that the orientation of $\partial q_{[\frac{1}{2},\frac{1}{2},\frac{1}{2}]}$ agrees with the orientations of $f_{[1,\frac{1}{2},\frac{1}{2}]}$, $f_{[\frac{1}{2},1,\frac{1}{2}]}$ and $f_{[\frac{1}{2},\frac{1}{2},1]}$ and disagrees with $f_{[0,\frac{1}{2},\frac{1}{2}]}$, $f_{[\frac{1}{2},0,\frac{1}{2}]}$ and $f_{[\frac{1}{2},\frac{1}{2},0]}$.

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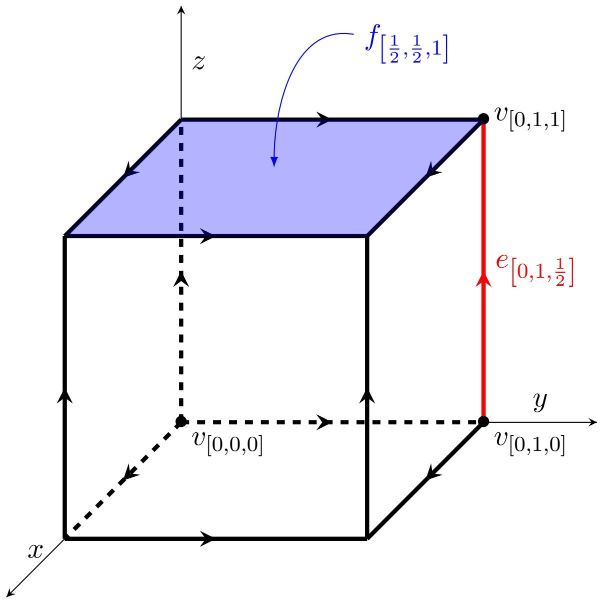

Figure 17. The cubic lattice with chosen orientations.

Let us now consider the crossed module $\mathbb{G}_{44} = (\Phi, \mathcal{E}, \Delta, \triangleright)$ with $\Phi \cong \mathcal{E} \cong \mathbb{Z}_4$, $\Delta(n) = 2n$ for $n \in \Phi$ and $m \triangleright n = (-1)^m n$ for $n \in \Phi$ and $m \in \mathcal{E}$. This is an example of a crossed module with nontrivial[10] $\ker(\Delta)$ and $\mathrm{coker}(\Delta)$. Furthermore, $\mathcal{E}$ acts non-trivially on Φ. However, it is still relatively simple, since $\mathrm{coker}(\Delta)$ is abelian and acts trivially on $\ker(\Delta)$.

Our present goal is to write down an explicit formula for the proposed Hamiltonian for the above system. We shall denote basis states in the Hilbert space $\mathcal{H}$ as $|\boldsymbol{m}, \boldsymbol{n}\rangle$, where $\boldsymbol{m} = \{m_e\}$ and $\boldsymbol{n} = \{n_f\}$ are collections of integers modulo four. An operator which for a fixed edge e shifts $m_e \to m_e + 1$ will be denoted by T_e. The definition of T_f is analogous. Furthermore, we let:

$$\mathsf{U}_e \, |\boldsymbol{m}, \boldsymbol{n}\rangle = e^{\frac{i\pi m_e}{2}} \, |\boldsymbol{m}, \boldsymbol{n}\rangle \, , \qquad \mathsf{U}_f \, |\boldsymbol{m}, \boldsymbol{n}\rangle = e^{\frac{i\pi n_f}{2}} \, |\boldsymbol{m}, \boldsymbol{n}\rangle \, . \tag{3.17}$$

More generally, we define U_γ and U_σ in the self-evident way.

Fake flatness constraint takes the form

$$\sum_{e \in \partial f} m_e = 2 n_f, \tag{3.18}$$

where summation is taken over all edges contained in the boundary of the face f, regardless of orientations. In other words, we restrict attention to states invariant under operators $\mathsf{U}_f^2 \prod_{e \in \partial f} \mathsf{U}_e$. Secondly, there is the constraint of invariance with respect to vertex Gauss' operators, which can be written in the form

$$\mathsf{G}_v = \left(\prod_{e \colon v = t(e)} \mathsf{T}_e \right) \left(\prod_{e \colon v = s(e)} \mathsf{T}_e^3 \right) \prod_{f \colon v = b(f)} \left(\mathsf{T}_f^2 \frac{1 - \mathsf{U}_f^2}{2} + \frac{1 + \mathsf{U}_f^2}{2} \right) . \tag{3.19}$$

Finally, there are edge Gauss' operators, which take the form $\prod_{f \colon e \in \partial f} \mathsf{T}_f^2$.

[10] In this case $\ker(\Delta) \cong \mathrm{coker}(\Delta) \cong \mathbb{Z}_2$. Moreover, the Postnikov class, whose definition is given in the appendix C.3, is the nonzero element of $H^3(B\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

It can be shown that operators W_{f,ν_f} satisfying all conditions given in subsection 3.1 are essentially uniquely determined to be of the form

$$\mathsf{W}_{f,\nu_f} = \nu_{f,0}\left(1 - \mathsf{T}_f^2\right), \qquad \text{with some } \nu_{f,0} > 0. \tag{3.20}$$

On the other hand, the operators V_{e,μ_e} are given by

$$\mathsf{V}_{e,\mu_e} = \mu_{e,1}\left(2 - \mathsf{T}_e^2 \prod_{f\colon e\in\partial f}\left(\mathsf{T}_f + \mathsf{T}_f^3\right)\right), \qquad \text{with some } \mu_{e,1} > 0. \tag{3.21}$$

Imposing further translational invariance of the system we are forced to put all parameters $\nu_{f,0}$ and $\mu_{e,1}$ to be constant, i.e. independent of f and e, respectively. Therefore we have

$$\mathsf{H}_W = \nu\sum_f\left(1 - \mathsf{T}_f^2\right), \qquad \mathsf{H}_V = \mu\sum_e\left(2 - \mathsf{T}_e^2\prod_{f\colon e\in\partial f}\left(\mathsf{T}_f + \mathsf{T}_f^3\right)\right) \tag{3.22}$$

with some $\mu, \nu > 0$.

There is some freedom in the definition of H_A. One good choice is given by

$$\mathsf{H}_A = \eta\sum_f\left(2 - \mathsf{U}_{\partial f} - \mathsf{U}_{\partial f}^3\right), \qquad \text{with } \eta > 0. \tag{3.23}$$

Next we consider a ball q and denote by f_1, f_2 and f_3 faces with $b(f_i) = b(q)$ (they are necessarily contained in ∂q). The three remaining faces of q will be denoted by f_j, with $j = 4, 5, 6$. For each of these three faces we take e_j be the edge such that $s(e_j) = b(q)$ and $t(e_j) = b(f_j)$. With this notation we have:

$$\mathsf{U}_{\partial q} = \prod_{j=4}^{6}\mathsf{U}_{e_j^{-1}\triangleright f_j}\prod_{i=1}^{3}\mathsf{U}_{f_i}^\dagger, \tag{3.24}$$

where $\mathsf{U}_{e_j^{-1}\triangleright f_j}$ can be expressed in terms of elementary U_e and U_f operators as

$$\mathsf{U}_{e_j^{-1}\triangleright f_j} = \mathsf{U}_f\frac{1 + \mathsf{U}_e^2}{2} + \mathsf{U}_f^3\frac{1 - \mathsf{U}_e^2}{2}. \tag{3.25}$$

Hamiltonian H_B is essentially uniquely determined to be

$$\mathsf{H}_B = \theta\sum_q\left(1 - \mathsf{U}_{\partial q}\right), \qquad \text{with } \theta > 0. \tag{3.26}$$

3.3 Symmetries

We will now describe symmetries of our model. Firstly, any field configuration determines a flat gauge field $\overline{\boldsymbol{\epsilon}}$ valued in $\mathrm{coker}(\Delta)$. Such field is described up to gauge transformations by an element $[\overline{\boldsymbol{\epsilon}}] \in \mathrm{Hom}(\pi_1(X;*), \mathrm{coker}(\Delta))/\!/\mathrm{coker}(\Delta)$, with double slash denoting the quotient with respect to a group action (in this case given by conjugation), and $*$ being an arbitrarily chosen base point in X_0. One may regard $[\overline{\boldsymbol{\epsilon}}]$ as an essentially classical quantity, because it is unchanged by the action of all operators introduced in our model thus far. In particular, the subspace of $\mathcal{H}$ spanned by all $|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle$ corresponding to a single $[\overline{\boldsymbol{\epsilon}}]$ is H-invariant.

Secondly, there exist so-called electric symmetries. In order to introduce them, we need to define the following two groups: $\mathcal{E}_0$ is the subgroup of $\mathcal{E}$ consisting of all elements which commute with the whole $\mathcal{E}$ and act trivially on Φ, while Φ_0 is the subgroup of $\ker(\Delta)$ of all elements invariant under the action of whole $\mathcal{E}$.

Now let $\boldsymbol{\zeta} = \{\zeta_e\}$ be a collection of elements of $\mathcal{E}_0$ such that[11] $\zeta_{\partial f} = 1$ for each f. We define an operator $\mathsf{L}_1(\boldsymbol{\zeta})$ by the formula

$$\mathsf{L}_1(\boldsymbol{\zeta})|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = |\{\zeta_e\, \epsilon_e\}, \boldsymbol{\varphi}\rangle. \tag{3.27}$$

It is straightforward to check that this preserves fake flatness and that $\mathsf{L}_1(\boldsymbol{\zeta})$ commutes with H. Now suppose that $\boldsymbol{\zeta}$ is of the special form $\zeta_e = \lambda_{t(e)}\lambda_{s(e)}^{-1}$ for some collection $\boldsymbol{\lambda} = \{\lambda_v\}$ valued in $\mathcal{E}_0$. In this case we have $\mathsf{L}_1(\boldsymbol{\zeta}) = \mathsf{G}(\boldsymbol{\lambda})$, which acts trivially on physical states.

We conclude from the preceding discussion that operators L_1 define a representation of the cohomology group $H^1(X, \mathcal{E}_0)$ on the space of physical states. This is a 1-form symmetry with symmetry group $\mathcal{E}_0$.

Secondly, let $\boldsymbol{\kappa} = \{\kappa_f\}$ be a collection of elements of Φ_0 such that[12] $\kappa_{\partial q} = 1$ for every ball q. Then we can introduce

$$\mathsf{L}_2(\boldsymbol{\kappa})|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = |\boldsymbol{\epsilon}, \{\kappa_f\, \varphi_f\}\rangle. \tag{3.28}$$

Again, this operation preserves fake flatness and commutes with H. For a collection $\boldsymbol{\kappa}$ of the form $\kappa_f = \omega_{\partial f}$ for some $\boldsymbol{\omega} = \{\omega_e\}$ valued in Φ_0 we have $\mathsf{L}_2(\boldsymbol{\kappa}) = \mathsf{V}(\boldsymbol{\omega})$. Hence on the space of physical states a representation of $H^2(X, \Phi_0)$ is defined. This is a 2-form symmetry with symmetry group Φ_0.

The last type of symmetries we discuss here is related with automorphisms of crossed modules. An automorphism of $\mathbb{G}$ is a homomorphism $(E, F) : \mathbb{G} \to \mathbb{G}$ such that E and F are group automorphisms. An automorphism of $\mathbb{G}$ is said to be inner if it is of the form $E(\epsilon) = \xi\, \epsilon\, \xi^{-1}$, $F(\varphi) = \xi \triangleright \varphi$ for some ξ in $\mathcal{E}$.

Automorphisms of $\mathbb{G}$ form a group $\mathrm{Aut}(\mathbb{G})$, with inner automorphisms being its normal subgroup. The quotient group is called the group of outer automorphisms and denoted by $\mathrm{Out}(\mathbb{G})$. We remark that the name is potentially misleading, because elements of $\mathrm{Out}(\mathbb{G})$ are merely equivalence classes of automorphisms (typically there exists no embedding of $\mathrm{Out}(\mathbb{G})$ as a subgroup of $\mathrm{Aut}(\mathbb{G})$).

Now let (E, F) be an automorphism of $\mathbb{G}$. We define an operator $\mathsf{K}(E, F)$ by

$$\mathsf{K}(E, F)|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = |\{E(\epsilon_e)\}, \{F(\varphi_f)\}\rangle. \tag{3.29}$$

Clearly this defines a representation of $\mathrm{Aut}(\mathbb{G})$ on $\mathcal{H}$. Now let us observe that for an inner automorphism given by an element $\xi \in \mathcal{E}$ we have

$$\mathsf{K}(E, F)|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = |\{\xi\, \epsilon_e\, \xi^{-1}\}, \{\xi \triangleright \varphi\}\rangle = \mathsf{G}(\{\xi_v\})|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle \tag{3.30}$$

[11] ζ_γ is defined for arbitrary path γ by demanding that $\zeta_{\gamma\gamma'} = \zeta_\gamma \zeta_{\gamma'}$ whenever $s(\gamma) = t(\gamma')$.

[12] Element $\kappa_{\partial q}$ is defined in terms of the κ_f in the same way as $\varphi_{\partial q}$ is defined in terms of φ_f. It does not depend on $\boldsymbol{\epsilon}$ because elements κ_f are $\mathcal{E}$-invariant. Similar notations will be used in the remaining part of this subsection without further explanations.

with the constant collection $\xi_v = \xi$ for every v. As this is a gauge transformation, $\mathsf{K}(E, F)$ acts trivially on physical states in this case. Hence on the space of physical states a representation of the group $\mathrm{Out}(\mathbb{G})$ is defined. Hamiltonian H may or may not be invariant under the action of these transformations, depending on the choice of "coupling constant" functions $\{\mu_e\}$, $\{\nu_f\}$, $\{\eta_f\}$ and $\{\theta_q\}$ in its definition. It is always possible to choose these functions so that whole $\mathrm{Out}(\mathbb{G})$ is realized as a global symmetry group.

3.4 Vacuum states

One of the most interesting goals in the study of models described by hamiltonians $\mathsf{H} = \mathsf{H}_{\mathrm{M}} + \mathsf{H}_{\mathrm{E}}$ would be to describe their possible phases. We will now make a small step in this direction by describing the space of ground states of H in various limits in which diagonalization can be performed exactly. In each case we have found that the lowest energy subspace:

- is the space of states of a certain topological field theory,
- admits a basis whose elements are in one-to-one correspondence with homotopy classes of maps from X to some other space.

These results are summarized in the table 1. All proofs are given in the remainder of this section. For each hamiltonian we provide a more explicit description of the basis ground states, not involving classifying spaces.

We speculate that some features found in the discussed limits may be generic for certain regions in the phase diagram of our model:

- Ground states of H_{E} are characterized by strong fluctuations of holonomies. Similar behaviour is expected to be exhibited also by ground states of the full hamiltonian in the regime in which H_{E} dominates over H_{M}. Such phase, if it indeed exists, would likely be characterized by an area law for 1-holonomies and a volume law for 2-holonomies.
- The putative phase approximately described by ground states of H_{AW} would be characterized by a perimeter law for 1-holonomies and a volume law for 2-holonomies.
- For ground states of H_{M} slightly perturbed by the electric hamiltonian H_{E}, we expect a perimer law for 1-holonomies and an area law for 2-holonomies.
- In a phase continuously connected to dynamics of H_{BV} we expect an area law for 1-holonomies as well as for 2-holonomies.

This will be further corroborated by the discussion in subsection 3.5, where we consider certain still simple, but already not purely topological limits of our model. They are shown to reduce to more standard purely 1-form or 2-form gauge theories, which are believed to exhibit behaviour consistent with the description above.

Let us start by considering the ground states of H_{E}. We will minimize every term in (3.10) at the same time, which clearly minimizes the whole H_{E}. First let us observe that for every face f we have a representation of the group $\ker(\Delta)$ by operators $\mathsf{W}_f(\chi)$. It is

Hamiltonian of the model	Basis of ground states
$\mathsf{H}_E = \mathsf{H}_V + \mathsf{H}_W$	$[X, B\mathrm{coker}(\Delta)]$
$\mathsf{H}_{AW} = \mathsf{H}_A + \mathsf{H}_W$	$[X, B\mathcal{E}]$
$\mathsf{H}_M = \mathsf{H}_A + \mathsf{H}_B$	$[X, B\mathbb{G}']$
$\mathsf{H}_{BV} = \mathsf{H}_B + \mathsf{H}_V$	$[X, B\mathbb{G}]$

Table 1. The ground states for four models described by integrable hamiltonians containing two out of four terms of H. Here $[X, Y]$ is the set of homotopy classes of maps $X \to Y$. For the first two entries, the relevant spaces Y are classifying spaces of groups. In the last two entries classifying spaces of crossed modules are meant. $\mathbb{G}'$ is the crossed module consisting of the trivial homomorphism $\ker(\Delta) \to \mathcal{E}$ and action of $\mathcal{E}$ on $\ker(\Delta)$ inherited from the crossed module $\mathbb{G}$.

possible to diagonalize all of them at the same time. Eigenvectors are labeled by element $\widehat{\chi}$ of the Pontryagin dual of $\ker(\Delta)$, i.e. the group $\widehat{\ker(\Delta)}$ of homomorphisms $\ker(\Delta) \to \mathrm{U}(1)$. Such eigenvector, here labeled by $|\widehat{\chi}\rangle$, satisfies

$$\mathsf{W}_f(\chi)|\widehat{\chi}\rangle = \widehat{\chi}(\chi)|\widehat{\chi}\rangle \qquad \text{for every } \chi \in \ker(\Delta). \tag{3.31}$$

It then follows that we have

$$\mathsf{W}_{f,\nu}|\widehat{\chi}\rangle = \left(\sum_{\chi} \nu(\chi)\widehat{\chi}(\chi)\right)|\widehat{\chi}\rangle. \tag{3.32}$$

The quantity in the parenthesis is, by definition, $\widehat{\nu}(\widehat{\chi})$ - the Fourier transform of ν evaluated at $\widehat{\chi}$. We have assumed that functions ν_f defining terms W_{f,ν_f} in H_{E} are such that $\widehat{\nu_f}(\widehat{\chi}) \geq 0$, with the equality if and only if $\widehat{\chi} = 1$. Then zero is the smallest eigenvalue of W_{f,ν_f} and one has $\mathsf{W}_{f,\nu_f}|\widehat{\chi}\rangle = 0$ only for $\widehat{\chi} = 1$. This means that vectors minimizing H_{E} are invariant with respect to all $\mathsf{W}(\boldsymbol{\chi})$. An analogous analysis, involving Fourier analysis for the non-abelian group (see e.g. [53, Part II]) Φ instead of Pontryagin duality, shows that they have to be invariant also with respect to all $\mathsf{V}(\boldsymbol{\psi})$. Finally, we require also invariance with respect to Gauss' operators $\mathsf{G}(\boldsymbol{\xi})$. States satisfying all these requirements may be obtained by summing $|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle$ over an orbit of the group generated by all vertex, edge and plaquette transformations, say

$$\sum_{\boldsymbol{\xi},\boldsymbol{\psi},\boldsymbol{\chi}} \mathsf{G}(\boldsymbol{\xi})\mathsf{V}(\boldsymbol{\psi})\mathsf{W}(\boldsymbol{\chi})|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle \tag{3.33}$$

for some reference $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$. The next step is to understand the space of orbits.

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It is clear that two configurations with the same $\boldsymbol{\epsilon}$ are related by a plaquette transformation. Furthermore, for two configurations with the same $\overline{\boldsymbol{\epsilon}}$ one can perform an edge transformation to make $\boldsymbol{\epsilon}$ equal. Collection $\overline{\boldsymbol{\epsilon}}$ itself is not changed by edge and plaquette transformations, but it transforms with respect to vertex transformations in the way usual for a gauge field.

We conclude that there is a basis of ground states of H_{E} indexed by elements of $\mathrm{Hom}(\pi_1(X;*), \mathrm{coker}(\Delta))//\mathrm{coker}(\Delta)$. Distinct ground states may be distinguished by values of 1-holonomies along nontrivial loops in X. In other words, we have found the space of states of a topological gauge theory with gauge group $\mathrm{coker}(\Delta)$.

Secondly, we discuss the space of ground states of H_{AW}. It admits a basis consisting of vectors of the form

$$\sum_{\boldsymbol{\xi},\boldsymbol{\varphi}} \mathsf{G}(\boldsymbol{\xi})|\boldsymbol{\epsilon},\boldsymbol{\varphi}\rangle, \tag{3.34}$$

where $\boldsymbol{\epsilon}$ is any collection with $\epsilon_{\partial f}=1$ for each f. The sum over $\boldsymbol{\varphi}$ runs over collections with φ_f in $\ker(\Delta)$, by fake flatness. Distinct vectors of the form (3.34) are labeled by elements of $\mathrm{Hom}(\pi_1(X;*),\mathcal{E})//\mathcal{E}$ determined by $\boldsymbol{\epsilon}$. Hence we find the space of states of a topological gauge theory with gauge group $\mathcal{E}$.

Next we consider the ground states of H_{M}. This is facilitated by the fact that holonomy operators act diagonally. To minimize all terms in (3.14) at the same time we have to restrict attention to configurations $(\boldsymbol{\epsilon},\boldsymbol{\varphi})$ satisfying flatness conditions

$$\epsilon_{\partial f}=1 \qquad \text{for every face } f, \tag{3.35a}$$

$$\varphi_{\partial q}=1 \qquad \text{for every ball } q. \tag{3.35b}$$

The first condition implies that each φ_f is in $\ker(\Delta)$, by fake flatness. Besides these constraints, only gauge invariant states are allowed. Such state may be constructed by summing over the gauge orbit of some configuration $(\boldsymbol{\epsilon},\boldsymbol{\varphi})$ satisfying (3.35):

$$|[\boldsymbol{\epsilon},\boldsymbol{\varphi}]\rangle = \sum_{(\boldsymbol{\epsilon}',\boldsymbol{\varphi}')\sim(\boldsymbol{\epsilon},\boldsymbol{\varphi})} |\boldsymbol{\epsilon}',\boldsymbol{\varphi}'\rangle, \tag{3.36}$$

where we write $(\boldsymbol{\epsilon}',\boldsymbol{\varphi}')\sim(\boldsymbol{\epsilon},\boldsymbol{\varphi})$ if $(\boldsymbol{\epsilon}',\boldsymbol{\varphi}')$ and $(\boldsymbol{\epsilon},\boldsymbol{\varphi})$ are related by a gauge transformation. Thus there is a basis of the space of ground states whose elements are in one-to-one correspondence with gauge orbits of configurations $(\boldsymbol{\epsilon},\boldsymbol{\varphi})$ subject to conditions (3.35). We will now describe this space of orbits.

An admissible collection $\boldsymbol{\epsilon}$ defines a flat gauge field on X valued in $\mathcal{E}$. For every conjugacy class of homomorphisms $\pi_1(X;*)\to\mathcal{E}$ we focus on one representative $\boldsymbol{\epsilon}$. Having fixed $\boldsymbol{\epsilon}$, we consider the allowed $\boldsymbol{\varphi}$. They have to satisfy (3.35b). Furthermore, we have to identify collections related by

$$\varphi_f' = \psi_{\partial f}^{(\epsilon)} \varphi_f \tag{3.37}$$

for any collection $\boldsymbol{\psi}$ of elements of $\ker(\Delta)$. We note the fact that elements $\psi_{\partial f}^{(\epsilon)}$ actually depend on $\boldsymbol{\epsilon}$ only through $\overline{\boldsymbol{\epsilon}}$, since the image of Δ acts trivially on $\ker(\Delta)$.

The space of equivalence classes of admissible collections $\boldsymbol{\varphi}$ is the twisted cohomology group $H^2(X,\ker(\Delta),\overline{\boldsymbol{\epsilon}})$, as recalled in the apppendix B. It is not true in general that distinct cohomology classes correspond to different gauge orbits. This is because there might exist vertex transformations $\boldsymbol{\xi}$ which preserve $\boldsymbol{\epsilon}$:

$$\xi_{t(e)}\,\epsilon_e\,\xi_{s(e)}^{-1}=\epsilon_e \qquad \text{for each edge } e. \tag{3.38}$$

This formula implies that ξ_* commutes with ϵ_γ for every loop based at $*$. Secondly, given such ξ_* it is possible to uniquely determine ξ_v for every vertex v from the above relation. In summary, the group $\mathrm{Stab}_V(\boldsymbol{\epsilon})$ of vertex transformations preserving $\boldsymbol{\epsilon}$ is isomorphic to the group of elements of $\mathcal{E}$ whose adjoint action preserves the homomorphism $\pi_1(X,*)\to\mathcal{E}$ determined by $\boldsymbol{\epsilon}$. It acts on $H^2(X,\ker(\Delta),\overline{\boldsymbol{\epsilon}})$ by (abelian) group homomorphisms, so the quotient space is also a group.

We conclude that the set of gauge orbits of flat configurations is a disjoint union of groups $H^2(X,\ker(\Delta),\overline{\boldsymbol{\epsilon}})//\mathrm{Stab}_V(\boldsymbol{\epsilon})$, with $\boldsymbol{\epsilon}$ running through a set of representatives of elements of $\mathrm{Hom}(\pi_1(X;*),\mathcal{E})//\mathcal{E}$. We remark that this is also the space of states of a topological gauge theory based on the crossed module $\mathbb{G}'$ which consists of the trivial homomorphism $\ker(\Delta)\to\mathcal{E}$ and action of $\mathcal{E}$ on $\ker(\Delta)$ inherited from $\mathbb{G}$. Indeed, condition $\epsilon_{\partial f}=1$ for each f is precisely the fake flatness constraint for $\mathbb{G}'$. Moreover groups, in which fields $\boldsymbol{\epsilon},\boldsymbol{\varphi}$ as well as gauge transformations are valued, coincide.

Vacuum states corresponding to non-equivalent $\boldsymbol{\epsilon}$ may always be distinguished by values of 1-holonomy operators along nontrivial loops. It is not always true that states with the same $\boldsymbol{\epsilon}$ but non-equivalent $\boldsymbol{\varphi}$ can be discriminated by evaluating 2-holonomies, as illustrated by one of examples in subsection 2.4.

Last, but not least, we consider the problem of minimization of H_{BV}. There exists a basis of ground states indexed by homotopy classes of maps $X\to B\mathbb{G}$. This implies that ground states of this hamiltonian form the space of states of the Yetter's model. This fact was discussed also in [40, 42]. For the sake of completeness we include a proof here. Furthermore, we give another description of the space of ground states.

In order to minimize operators $\mathsf{B}_{\partial q,\theta_q}$ we have to restrict attention to configurations obeying $\varphi_{\partial q}=1$ for every ball q. Given any such configuration $(\boldsymbol{\epsilon},\boldsymbol{\varphi})$ we obtain a ground state by forming the superposition

$$\sum_{\boldsymbol{\xi},\boldsymbol{\psi}}\mathsf{G}(\boldsymbol{\xi})\mathsf{V}(\boldsymbol{\psi})|\boldsymbol{\epsilon},\boldsymbol{\varphi}\rangle. \tag{3.39}$$

A basis of the space of ground states is formed by vectors of this form, one for each orbit of the group of vertex and edge transformations in the set of admissible configurations. The fact that these orbits are in one-to-one correspondence with homotopy classes of maps $X\to B\mathbb{G}$ has been reviewed in the appendix C.2. We proceed to give an alternative description of the set of orbits.

Since in the present analysis configurations related by edge transformations with arbitrary $\boldsymbol{\psi}$ are identified, the only invariant datum specified by $\boldsymbol{\epsilon}$ is the corresponding element of $\mathrm{Hom}(\pi_1(X;*),\mathrm{coker}(\Delta))//\mathrm{coker}(\Delta)$. For every element of this set we choose one representative $\overline{\boldsymbol{\epsilon}}$ and lift it to some $\boldsymbol{\epsilon}$. It is not always possible to choose $\boldsymbol{\epsilon}$ which is itself flat, as shown in examples in subsection 2.4.

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Next we consider the set of allowed $\boldsymbol{\varphi}$ for the given $\boldsymbol{\epsilon}$. As illustrated in subsection 2.4, for $\overline{\boldsymbol{\epsilon}}$ having nontrivial holonomies it may happen that no $\boldsymbol{\varphi}$ satisfying the flatness condition $\varphi_{\partial q} = 1$ exists. Let us consider the case in which some flat $\boldsymbol{\varphi}$ does exist. Then any other flat $\boldsymbol{\varphi}'$ is of the form

$$\varphi'_f = \chi_f \, \varphi_f \tag{3.40}$$

for some twisted cocycle $\boldsymbol{\chi}$ (see apppendix B). The more stringent condition that $\boldsymbol{\chi}$ is a twisted coboundary holds if and only if configurations $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ and $(\boldsymbol{\epsilon}, \boldsymbol{\varphi}')$ are related by an edge transformation with ψ_e in $\ker(\Delta)$ for each e. Therefore the set $\mathcal{F}(\boldsymbol{\epsilon})$ of equivalence classes of flat $\boldsymbol{\varphi}$ (with the given $\boldsymbol{\epsilon}$) modulo $\ker(\Delta)$-valued edge transformations is an affine space over $H^2(X, \ker(\Delta), \overline{\boldsymbol{\epsilon}})$. That is not the end of the story, because the group $\mathrm{Stab}_{V,E}(\boldsymbol{\epsilon})$ of combined vertex and edge transformations preserving $\boldsymbol{\epsilon}$ acts on $\mathcal{F}(\boldsymbol{\epsilon})$. We will now show that this action factors through the smaller group $\mathrm{Stab}_V(\overline{\boldsymbol{\epsilon}})$ of $\mathrm{coker}(\Delta)$-valued vertex transformations, i.e. collections $\overline{\boldsymbol{\xi}} = \{\overline{\xi}_v\}$ of elements of $\mathrm{coker}(\Delta)$ such that

$$\overline{\epsilon}_\gamma = \overline{\xi}_{t(\gamma)} \, \overline{\epsilon}_\gamma \, \overline{\xi}_{s(\gamma)}^{-1} \qquad \text{for every path } \gamma. \tag{3.41}$$

Transformations in $\mathrm{Stab}_{V,E}(\boldsymbol{\epsilon})$ are represented by operators of the form $\mathsf{G}(\boldsymbol{\xi})\mathsf{V}(\boldsymbol{\psi})$, where the pair $(\boldsymbol{\xi}, \boldsymbol{\psi})$ is such that

$$\epsilon_\gamma = \xi_{t(\gamma)} \, \Delta\left(\psi_\gamma^{(\epsilon)}\right) \, \epsilon_\gamma \, \xi_{s(\gamma)}^{-1} \qquad \text{for any path } \gamma. \tag{3.42}$$

Let us first consider a pair $(\boldsymbol{\xi}, \boldsymbol{\psi})$ such that in addition $\xi_v = \Delta\rho_v$ for each v. The discussion around equation (2.37) gives

$$\mathsf{G}(\boldsymbol{\xi})\mathsf{V}(\boldsymbol{\psi})|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = \mathsf{V}(\widetilde{\boldsymbol{\psi}})\mathsf{V}(\boldsymbol{\psi})|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle = \mathsf{V}(\widetilde{\boldsymbol{\psi}}\boldsymbol{\psi})|\boldsymbol{\epsilon}, \boldsymbol{\varphi}\rangle \tag{3.43}$$

for some Φ-valued collection $\widetilde{\boldsymbol{\psi}}$. By the definition of $\mathrm{Stab}_{V,E}(\boldsymbol{\epsilon})$, we must have $\widetilde{\psi}_e \psi_e \in \ker(\Delta)$ for each e. Thus the action of $(\boldsymbol{\xi}, \boldsymbol{\psi})$ on the set of allowed $\boldsymbol{\varphi}$ for the given $\boldsymbol{\epsilon}$ reduces to a $\ker(\Delta)$-valued edge transformation. Hence $(\boldsymbol{\xi}, \boldsymbol{\psi})$ acts trivially on the set $\mathcal{F}(\boldsymbol{\epsilon})$.

Next let us observe that the map $\mathrm{Stab}_{V,E}(\boldsymbol{\epsilon}) \ni (\boldsymbol{\xi}, \boldsymbol{\psi}) \mapsto \overline{\boldsymbol{\xi}} \in \mathrm{Stab}_V(\overline{\boldsymbol{\epsilon}})$ is a homomorphism. Preceding discussion shows that its kernel acts trivially on $\mathcal{F}(\boldsymbol{\epsilon})$, so to complete the proof it is sufficient to show that this homomorphism is surjective. Thus we choose some $\overline{\boldsymbol{\epsilon}}$ obeying (3.41) and lift it to an $\mathcal{E}$-valued collection $\boldsymbol{\epsilon}$ arbitrarily. By construction, we have that for each path γ the element

$$\mu_\gamma = \xi_{t(\gamma)}^{-1} \, \epsilon_\gamma \, \xi_{s(\gamma)} \, \epsilon_\gamma^{-1} \tag{3.44}$$

belongs to $\mathrm{im}(\Delta)$. Directly from the definition of μ_γ we have that

$$\mu_{\gamma'\gamma} = \mu_{\gamma'} \, \epsilon_{\gamma'} \, \mu_\gamma \, \epsilon_{\gamma'}^{-1} \tag{3.45}$$

is satisfied for any composite path $\gamma'\gamma$. Now for every edge e we choose some ψ_e such that $\mu_e = \Delta\psi_e$. Then μ_γ coincides with $\Delta\psi_\gamma^{(\epsilon)}$ whenever γ is a single edge, and furthermore these two collections satisfy the same composition rule for concatenated paths. Thus $\mu_\gamma = \Delta\psi_\gamma^{(\epsilon)}$ for every γ. Plugging this into (3.44) we obtain

$$\xi_{t(\gamma)} \, \Delta\psi_\gamma^{(\epsilon)} \, \epsilon_\gamma \, \xi_{s(\gamma)}^{-1} = \epsilon_\gamma, \tag{3.46}$$

and hence the claim is proven.

In summary, vectors of the form (3.39) form a basis of ground states of H_{BV}. They can be labeled by elements of the disjoint union of sets $\mathcal{F}(\boldsymbol{\epsilon})/\!/\mathrm{Stab}_V(\overline{\boldsymbol{\epsilon}})$ with $\boldsymbol{\epsilon}$ running through representatives of elements of $\mathrm{Hom}(\pi_1(X;*), \mathrm{coker}(\Delta))/\!/\mathrm{coker}(\Delta)$.

We are now ready to deduce that the space of ground states of H_{BV}, as well as the space of states invariant under all vertex and edge transformations, but not necessarily with flat $\boldsymbol{\varphi}$, depends on $\mathbb{G}$ only through its weak equivalence class. Clearly it is sufficient to consider a weak isomorphism $(E, F) : \mathbb{G} \to \mathbb{G}' = (\mathcal{E}', \Phi', \Delta', \triangleright')$. We let T be the map which sends a $\mathbb{G}$-valued configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ to $(E(\boldsymbol{\epsilon}), F(\boldsymbol{\varphi}))$, where $E(\boldsymbol{\epsilon}) = \{E(\epsilon_e)\}$ and $F(\boldsymbol{\varphi}) = \{F(\varphi_f)\}$. This intertwines between $\mathbb{G}$ and $\mathbb{G}'$-valued vertex and edge transformations, so there is an induced mapping on the space of orbits. The latter is in both cases a disjoint union of subsets labeled by

$$\mathrm{Hom}(\pi_1(X;*), \mathrm{coker}(\Delta))/\!/\mathrm{coker}(\Delta) \cong \mathrm{Hom}(\pi_1(X;*), \mathrm{coker}(\Delta'))/\!/\mathrm{coker}(\Delta'),$$

where we used the fact that $\overline{E} : \mathrm{coker}(\Delta) \to \mathrm{coker}(\Delta')$ is an isomorphism. Mapping T preserves this decomposition. Thus it is sufficient to consider configurations of the form $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ and $(E(\boldsymbol{\epsilon}), \boldsymbol{\varphi}')$ for one $\boldsymbol{\epsilon}$, for now with no constraint on $\boldsymbol{\varphi}$. Let $\mathcal{C}(\boldsymbol{\epsilon})$ and $\mathcal{C}'(E(\boldsymbol{\epsilon}))$ be the sets of all allowed $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}'$ modulo $\ker(\Delta)$- (resp. $\ker(\Delta')$-)valued edge transformations. They are affine over $H^2(X_2, \ker(\Delta), \overline{\boldsymbol{\epsilon}}) \cong H^2(X_2, \ker(\Delta'), \overline{E}(\overline{\boldsymbol{\epsilon}}))$, because a flat configuration on X_2 is the same as an arbitrary configuration on X (constraint $\varphi_{\partial q} = 1$ for every ball q being vacuous if balls are absent). The map T intertwines between the affine structures, so we have $\mathcal{C}(\boldsymbol{\epsilon}) \cong \mathcal{C}'(E(\boldsymbol{\epsilon}))$. Furthermore, we clearly have $\mathrm{Stab}_V(\overline{\boldsymbol{\epsilon}}) \cong \mathrm{Stab}_V(\overline{E}(\overline{\boldsymbol{\epsilon}}))$, and again T preserves actions of these groups. Thus

$$\mathcal{C}(\boldsymbol{\epsilon})/\!/\mathrm{Stab}_V(\overline{\boldsymbol{\epsilon}}) \cong \mathcal{C}'(E(\boldsymbol{\epsilon}))/\!/\mathrm{Stab}_V(E(\overline{\boldsymbol{\epsilon}})),$$

which proves that T is a bijection. Finally, let us observe that $F(\boldsymbol{\varphi})_{\partial q} = \overline{F}(\varphi_{\partial q})$, since $\varphi_{\partial q} \in \ker(\Delta)$. Since $\overline{F}$ is an isomorphism, this implies that flatness of $F(\boldsymbol{\varphi})$ is equivalent to flatness of $\boldsymbol{\varphi}$. Hence T is bijective also after restricting to flat configurations.

Another interesting point to be raised here is that there is an explicit topological criterion to determine when the set $\mathcal{F}(\boldsymbol{\epsilon})$ is nonempty. Here we give a short summary, with a more detailed description postponed to the appendix C.3. Field $\overline{\boldsymbol{\epsilon}}$ determines (up to homotopy) a map $h_{\overline{\boldsymbol{\epsilon}}}$ from X to $B\mathrm{coker}(\Delta)$, the classifying space of the group $\mathrm{coker}(\Delta)$. There is a distiniguished twisted cohomology class β on $B\mathrm{coker}(\Delta)$, called the Postnikov class. The set $\mathcal{F}(\boldsymbol{\epsilon})$ is nonempty if and only if the pullback $h_{\overline{\boldsymbol{\epsilon}}}^* \beta$ is the trivial cohomology class on X.

We close this section with the remark that it has been shown in [29] that the topological field theory describing ground states of H_{VB} may be formulated using fields valued in groups $\ker(\Delta)$ and $\mathrm{coker}(\Delta)$ only. In this approach crossed modules do not have to be invoked explicitly. One has to merely specify the action of $\mathrm{coker}(\Delta)$ on $\ker(\Delta)$ and the Postnikov class β. These are precisely the data that determine the crossed module up to weak isomorphisms [54], in accord with the fact that the model possesses weak isomorphism invariance.

3.5 A peek at dynamics

In this subsection we discuss models described by hamiltonians in which three out of four terms of H are present. See figure 18 for an illustration of the four possibilities. In each

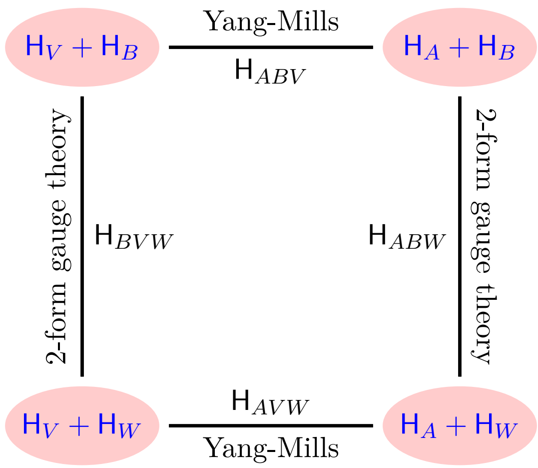

Figure 18. A diagram representing four possible models with hamiltonians consisting of three out of four terms.

case dynamics reduces to that of some simpler theory. Therefore we can understand the dynamics generated by H along the boundary of its phase diagram.

Several topological aspects of our models have been discussed in subsection 3.4. In this step we would like to focus instead on local dynamics. Therefore we assume now a topologically trivial situation, i.e. that the first two homotopy groups of X vanish. In this case we can always fix gauge $\overline{\epsilon_e} = 1$. In other words, the $\boldsymbol{\epsilon}$ field can be regarded as valued in $\mathrm{im}(\Delta)$. True physical states may be obtained in the end by summing over vertex transformations. Thus in the further analysis it is necessary to explicitly take into account only those vertex transformations $\boldsymbol{\xi}$ which preserve the gauge condition $\overline{\epsilon_e} = 1$, i.e. those with constant $\overline{\boldsymbol{\xi}}$.

Let us begin with the case $\mathsf{H}_{ABW} = \mathsf{H}_A + \mathsf{H}_B + \mathsf{H}_W$. Term H_A commutes with the other two, so it may be minimized exactly.[13] Therefore we may restrict attention to field configurations with $\epsilon_{\partial f} = 1$ for each f. Each gauge equivalence class of fields with this property admits a representative with $\epsilon_e = 1$ for each edge e. For these representatives the fake flatness constraint implies that $\varphi_f \in \ker(\Delta)$. As a result, the only physical degree of freedom is a $\ker(\Delta)$-valued 2-form field. Residual gauge freedom consists of transformations of two types: edge transformations valued in $\ker(\Delta)$, which play the role of standard gauge transformations for the 2-form field, and vertex transformations with constant ξ. Explicitly, the latter acts according to the formula $\varphi_\sigma \longmapsto \xi \triangleright \varphi_\sigma$ for every σ. From the point of view of the 2-form theory this is a global symmetry. Summarizing, the ground states of H_{ABW} coincide with ground states of a 2-form gauge theory valued in $\ker(\Delta)$, restricted to the singlet sector of a certain global symmetry.

For the hamiltonian H_{AVW}, the analysis is analogous. Field φ is effectively removed by exactly minimizing H_W, which enforces that for any $\boldsymbol{\epsilon}$ all configurations $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ allowed by fake flatness enter with an equal amplitude. The final conclusion is that ground states

[13]Strictly speaking, it could happen that an eigenvector of $\mathsf{H}_B + \mathsf{H}_W$ to a much lower eigenvalue could be found in excited subspaces of H_A. Thus presented analysis is valid exactly only under the additional assumption that H_A dominates over the other two terms.

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are the same as in a Yang-Mills theory with gauge group $\mathrm{im}(\Delta)$, restricted to the singlet sector of a global symmetry.

Next, consider the model with hamiltonian H_{ABV}. In this case we impose the constraint $\varphi_{\partial q} = 1$ for each q. There exists a unique such $\boldsymbol{\varphi}$, up to edge transformations valued in $\ker(\Delta)$, for every $\boldsymbol{\epsilon}$. Therefore the field $\boldsymbol{\varphi}$ is effectively removed from the theory. In the end we obtain the singlet sector of lattice Yang-Mills theory with gauge group $\mathrm{im}(\Delta)$, as in the case of H_{AVW}.

It remains to analyze the theory with H_{BVW} as a hamiltonian. In this case we minimize exactly the H_V term. Therefore ground states may be written as superpositions of states of the form

$$\sum_{\boldsymbol{\psi}} \mathsf{V}(\boldsymbol{\psi})|\mathbf{1}, \boldsymbol{\varphi}\rangle = \sum_{\boldsymbol{\psi}} |\{\Delta\psi_e\}, \{\psi_{\partial f}\, \varphi_f\}\rangle, \tag{3.47}$$

which are labeled by collections $\boldsymbol{\varphi}$ valued in $\ker(\Delta)$, modulo 2-form gauge transformations $\varphi_f \mapsto \psi_{\partial f}\, \varphi_f$ with $\ker(\Delta)$-valued $\boldsymbol{\psi}$. Thus we obtain the space of states of a 2-form gauge theory. Vertex gauge transformations with $\boldsymbol{\xi}$ valued in $\mathrm{im}(\Delta)$ act trivially, because they reduce to edge transformations, which were already taken care of. There remains only the condition of invariance with respect to vertex transformations with constant $\boldsymbol{\xi}$, which again can be interpreted as a global symmetry.

Finally, we would like to emphasize that, in spite of the preceding discussion, models found on opposite edges of the diagram on figure 18 are not identical. They differ in their global properties once we start considering spaces X with nontrivial homotopy groups. Firstly, let us compare hamiltonians H_{ABV} and H_{AVW}. In the first case low-lying states have flat $\boldsymbol{\varphi}$, but can be distinguished by 2-holonomies along non-contractible spheres in X. There is a possibility of ground state degeneracy due to existence of several non-equivalent flat $\boldsymbol{\varphi}$ for a given $\boldsymbol{\epsilon}$. Thus the 2-form electric symmetry may be broken. On the other hand for the hamiltonian H_{AVW} field $\boldsymbol{\varphi}$ is effectively absent. Since ground states are invariant under all W operators, the 2-form symmetry is unbroken. Comparison of H_{ABW} and H_{BVW} is similar. In the former case fields $\boldsymbol{\epsilon}$ are flat, but they may still have nontrivial 1-holonomies. Thus the 1-form electric symmetry may be broken. On the other hand for ground states of H_{BVW} holonomies of $\boldsymbol{\epsilon}$ are undefined, since they are not invariant with respect to edge transformations (which are symmetries of the states).

Acknowledgments

LH would like to thank Martin Roček and Rikard von Unge for a discussion which initiated the current project. We also thank A. Czarnecki for a discussion. BR was supported by the MNS donation for PhD students and young scientists N17/MNS/000040. The work of LH was supported by the TEAM programme of the Foundation for Polish Science co-financed by the European Union under the European Regional Development Fund (POIR.04.04.00-00-5C55/17-00).

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A Kernel and cokernel of ∂

Given a topological space A, subspace B and a base point $* \in B$, one has an exact sequence of groups [50, Thm. 4.3]

$$\pi_1(A;*) \longleftarrow \pi_1(B;*) \xleftarrow{\partial} \pi_2(A,B;*) \longleftarrow \pi_2(A;*) \longleftarrow \pi_2(B;*). \tag{A.1}$$

It follows that $\ker(\partial)$ may be identified with the quotient of $\pi_2(A;*)$ by the image of the homomorphism $\pi_2(B;*) \to \pi_2(A;*)$.

Furthermore, notice that if the map $\pi_1(B;*) \to \pi_1(A;*)$ is surjective, the cokernel of ∂ is isomorphic to $\pi_1(A;*)$. This is true in particular for $A = X$, $B = X_1$, by the cellular approximation theorem [50, Thm. 4.8]. Secondly, the universal cover a one-dimensional CW-complex is contractible. Thus $\pi_2(X_1;*)$ is trivial, so we have an identification $\ker(\partial) \cong \pi_2(X;*)$.

B Twisted cohomology

In this appendix we give a definition of twisted cohomology as it arises directly in calculations done in this paper. We refer to [55, p. 255–290] for a more complete treatment. We shall use relative homotopy groups $\pi_n(A,B;*)$ with any $n \geq 2$, as well as the action of $\pi_1(B;*)$ on these groups. Their definition is entirely analogous to the case $n=2$ and can be found e.g. in [50, p. 343]. They are abelian for $n \geq 3$. As for $n=2$, there is a homomorphism $\partial : \pi_n(A,B;*) \xrightarrow{\partial} \pi_{n-1}(B;*)$, whose kernel coincides with the image of the self-evident map $\pi_n(A;*) \to \pi_n(A,B;*)$. Furthermore, a map $A \to A'$ which takes B to $B' \subseteq A'$ induces a homomorphism $\pi_n(A,B;*) \to \pi_n(A',B';*)$, which is unchanged by homotopic deformations preserving the condition that B is mapped to B' at all intermediate stages. All that generalizes to a groupoid version $\pi_n(A,B;C)$, for which a whole set $C \subseteq B$ of base points is allowed, in a way analogous to the case $n=2$.

In our applications we need the above structure with $A = X_n$, $B = X_{n-1}$ and $C = X_0$. Thus $\pi_1(B;C) = \pi_1(X_1;X_0)$ if $n=2$ and $\pi_1(B;C) = \pi_1(X;X_0)$ for $n \geq 3$. Since the latter group is a quotient of $\pi_1(X_1;X_0)$, we have an action of $\pi_1(X_1;X_0)$ on $\pi_n(X_n,X_{n-1};X_0)$ in each case. Groups $\pi_n(X_n,X_{n-1};x)$ with $x \in X_0$ and $n \geq 3$ may be handled in practice using the fact [55, p. 212] that they are free $\pi_1(X;x)$-modules, with bases labeled by n-cells of X.

Now let us fix a group G, an abelian group K on which G acts by automorphisms and a homomorphism $\boldsymbol{\alpha} : \pi_1(X;X_0) \to G$. Thus for every path γ there is an endomorphism $k \mapsto \alpha_\gamma \triangleright k$ of K, trivial if γ is contractible in X. It obeys the composition law $\alpha_{\gamma'\gamma} = \alpha_{\gamma'}\,\alpha_\gamma$. In our applications we will mostly consider the case $G = \mathrm{coker}(\Delta)$ and $K = \ker(\Delta)$ for some crossed module $\mathbb{G}$, with $\boldsymbol{\alpha} = \overline{\boldsymbol{\epsilon}}$. This is not relevant for the discussion in this appendix.

By an $\boldsymbol{\alpha}$-twisted p-cochain on X valued in K we shall mean:

- $p=0$: collection of elements $\rho_v \in K$ labeled by vertices v,
- $p=1$: assignment of $\psi_\gamma \in K$ to every path γ, subject to the composition law $\psi_{\gamma'\gamma} = \psi_{\gamma'}\,(\alpha_{\gamma'} \triangleright \psi_\gamma)$ whenever $s(\gamma') = t(\gamma)$,

$$
\begin{tikzcd}
 & & \pi_{p+1}(X_{p+1}, X_p; X_0) \arrow[drr, dashed, "\partial"] \arrow[ddr, "\overline{\partial}"] & & \\
\pi_{p+1}(X_{p+1}; X_0) \arrow[urr, dashed] \arrow[rrrr, dashed] & & & & \pi_p(X_p; X_0) \arrow[dl, dashed] \\
 & \pi_{p+2}(X_{p+2}, X_{p+1}; X_0) \arrow[ul, dashed, "\partial"] \arrow[uur, "\overline{\partial}"] \arrow[rr] & & \pi_p(X_p, X_{p-1}; X_0) &
\end{tikzcd}
$$

Figure 19. The two colored maps factors through the dashed ones marked by the same colors. The composition depictured by the dashed black arrow is the trivial map.

- $p \geq 2$: homomorphism $\chi : \pi_p(X_p, X_{p-1}; X_0) \to K$ satisfying the equivariance condition $\chi_{\gamma \triangleright \tau} = \alpha_\gamma \triangleright \chi_\tau$.

The set of all p-cochains is a group, which we denote by $C^p(X, K, \boldsymbol{\alpha})$. Next we define a differential $\delta : C^p(X, K, \boldsymbol{\alpha}) \to C^{p+1}(X, K, \boldsymbol{\alpha})$ in the following way:

- $p = 0$: $(\delta\rho)_\gamma = \rho_{t(\gamma)} \left(\alpha_\gamma \triangleright \rho^{-1}_{s(\gamma)} \right)$,
- $p = 1$: $(\delta\psi)_\sigma = \psi_{\partial\sigma}$, where $\partial : \pi_2(X_2, X_1; X_0) \to \pi_1(X_1; X_0)$ is as defined in section 2.1,
- $p \geq 2$: $(\delta\chi)_\tau = \chi_{\overline{\partial}\tau}$, where $\overline{\partial} : \pi_{p+1}(X_{p+1}, X_p; X_0) \to \pi_p(X_p, X_{p-1}; X_0)$ is the composition of homomorphisms $\partial : \pi_{p+1}(X_{p+1}, X_p; X_0) \to \pi_p(X_p; X_0)$ and $\pi_p(X_p; X_0) \to \pi_p(X_p, X_{p-1}; X_0)$.

With this differential, $C^\bullet(X, K, \boldsymbol{\alpha})$ is a cochain complex, whose cohomology we denote by $H^\bullet(X, K, \boldsymbol{\alpha})$ and call the twisted cohomology. Another popular name is cohomology with local coefficients. To see that δ is nilpotent, first note that for a 0-cochain ρ we have $(\delta^2\rho)_\sigma = (\delta\rho)_{\partial\sigma} = \rho_{b(\sigma)} \left(\alpha_{\partial\sigma} \triangleright \rho^{-1}_{b(\sigma)} \right) = 1$, as $\alpha_{\partial\sigma} = 1$. For a p-cochain $\boldsymbol{\chi}$ with $p \geq 2$ we have $(\delta^2\chi)_\tau = \chi_{\overline{\partial}^2\tau}$. Homomorphism $\overline{\partial}^2$ fits in the commutative diagram shown on figure 19, so it factors through the (trivial) composition of two subsequent homomorphisms in the long exact sequence of relative homotopy groups [50, Thm. 4.3] of the pair (X_{p+1}, X_p). For $p = 1$ one needs triviality of $\partial\overline{\partial}$, for which an analogous reasoning applies.

Now let us assume that $l : Y \to X$ is a cellular map of CW-complexes. Given $\boldsymbol{\alpha} \in \mathrm{Hom}(\pi_1(X, X_0), G)$, its pullback $l^*\boldsymbol{\alpha} \in \mathrm{Hom}(\pi_1(Y, Y_0), G)$ is defined as the composition of $\boldsymbol{\alpha}$ with the pushforward map $\pi_1(Y, Y_0) \to \pi_1(X, X_0)$ induced by l. Furthermore, the pullback $l^* : C^\bullet(X, K, \boldsymbol{\alpha}) \to C^\bullet(Y, K, l^*\boldsymbol{\alpha})$ may be defined in an analogous way. It intertwines between the differentials, so there is an induced pullback map of cohomology $l^* : H^\bullet(X, K, \boldsymbol{\alpha}) \to H^\bullet(Y, K, l^*\boldsymbol{\alpha})$.

We close this appendix with a remark that twisted cohomology may be defined also without reference to a cell structure on X. They depend only on the topology of X

and another datum called a local system of abelian groups on X. The latter may be (non-canonically) encoded by a single abelian group K and a homomorphism $\pi_1(X;*) \to \mathrm{Aut}(K)$ for some base point $*$.

C Classifying spaces

Due to the length of this appendix, we divided it into several parts. In C.1 we recall the basic properties of classifying spaces of groups. Appendix C.2 is devoted to the definition and the proof of the fundamental property of classifying spaces of crossed modules, which relates field configurations on a space X valued in a crossed module $\mathbb{G}$ with maps $X \to B\mathbb{G}$. In C.3 we explain the relation of the so-called Postnikov class with the problem of constructing field configurations (or equivalently, maps to $B\mathbb{G}$). In C.4 we construct maps between classifying spaces corresponding to homomorphisms of crossed modules and obtain the corollary that weakly equivalent crossed modules have homotopy equivalent classifying spaces. A simple proof of existence of classifying spaces is given in C.5.

C.1 Classifying spaces of groups

We begin with a short review of the classifying space BG of a group G. One way to define it[14] is as a connected CW-complex with fundamental group G and trivial higher homotopy groups. It is well known [55, Thm. 7.1] that such space exists and is determined uniquely up to a homotopy equivalence. One may also assume that BG has exactly one 0-cell $*$, which we take as its base point.

We claim that gauge orbits of G-valued lattice gauge fields on X are in one-to-one correspondence with homotopy classes of maps $X_1 \to BG$. Flatness of a gauge field is equivalent to existence of an extension of the corresponding map to X_2. If this condition is satisfied, extending to the whole X is automatic, and furthermore this extension is unique up to homotopy. There is also a correspondence between flat gauge fields (rather than gauge equivalence classes) on X and homotopy classes of maps of pairs[15] $(X, X_0) \to (BG, *)$. Again, flatness condition may be lifted by considering maps $(X_1, X_0) \to (BG, *)$.

To prove the above claims, let us first note that any mapping $X \to BG$ is homotopic to one which sends the whole X_0 to $*$, by the homotopy extension property [50, p. 15] of the pair (X, X_0). Such map sends every edge of X to a loop in BG based at the base point $*$. As a result it determines a homomorphism $\pi_1(X_1, X_0) \to \pi_1(BG, *) \cong G$, i.e. a lattice gauge field on X. Two maps $h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}'}$ are homotopic if and only if they determine gauge-equivalent fields $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$. Indeed, constructing a homotopy between them amounts to constructing an extension to[16] $I \times X$ of the map $\{0,1\} \times X \to BG$ given by $h_{\boldsymbol{\alpha}}$ and $h_{\boldsymbol{\alpha}'}$, respectively, on $\{0\} \times X$ and $\{1\} \times X$. This can be done iteratively, cell-by-cell.

[14] There is a more general notion of a classifying space of a topological group [56], for which this definition is not suitable. Here only discrete groups are considered.

[15] Map of pairs $(X, Y) \to (X', Y')$ with $Y \subseteq X$ and $Y' \subseteq X'$ is a map $X \to X'$ which takes Y into Y'. Definition of homotopy classes of maps of pairs allows only homotopies for which Y is mapped to Y' at all times. Maps of triples and their homotopy classes are defined analogously.

[16] Here I is the unit interval.

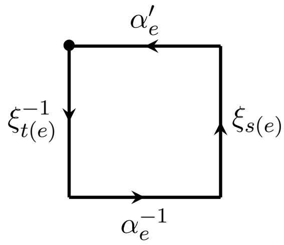

Figure 20. Extension problem encountered in the construction of a homotopy between two maps $X \to BG$ cell by cell. The bold dot is the chosen base point of the square.

First we consider 1-cells, which are of the form $I \times \{v\}$ with v - vertices of X. These can be mapped to any loops in BG, which determine elements $\xi_v \in \pi_1(BG, *)$. Next we extend through 2-cells, which are products of I and edges of X. Considering an edge e, we arrive at the problem of extending to the whole square the map on the boundary depictured on figure 20. This is possible if and only if the boundary map is null-homotopic, i.e. if $\alpha'_e \, \xi_{s(e)} \, \alpha_e^{-1} \, \xi_{t(e)}^{-1} = 1$ in G. In other words, if $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}$ are gauge equivalent, we have to choose $\boldsymbol{\xi}$ in the previous step which is a gauge transformation from $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}'$. Afterwards one has to extend through higher cells. This is always possible since higher homotopy groups of BG vanish. Thus $h_{\boldsymbol{\alpha}}$ and $h_{\boldsymbol{\alpha}'}$ are homotopic. If $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are not gauge equivalent, it is impossible to extend the map through 2-cells regardless of the choice of an extension through 1-cells. Hence $h_{\boldsymbol{\alpha}}$ and $h_{\boldsymbol{\alpha}'}$ are not homotopic.

We still have to determine which gauge fields can be realized by some map to BG. On the 1-skeleton of X we can realize any gauge field, simply by constructing the corresponding map cell-by-cell. An obstruction arises if one attempts to extend the map from X_1 to X_2. Concretely, extension over a face f is possible if and only if the bounding loop is sent to a trivial loop in BG, i.e. if $\alpha_{\partial f} = 1$. Thus a map $h : X_1 \to BG$ extends to X_2 if and only if the corresponding gauge field is flat. Further extension from X_2 to X is unobstructed, again because higher homotopy groups of BG are trivial.

The only part that remains to be proven is the one concerning homotopy classes of pairs $(X, X_0) \to (BG, *)$. Such homotopy class determines a homomorphism $\pi_1(X_1, X_0) \to G$. We already know that every homomorphism is realized by some homotopy class. Furthermore, in the construction of a homotopy between two maps determined by the same homomorphism we may take $\xi_v = 1$ for every v, and hence the homotopy may be taken to be stationary on $I \times X_0$.

Being done with the proof, notice that there exists a distinguished G-valued gauge field $\boldsymbol{\iota}$ on BG, corresponding to the tautological (identity) homomorphism $\pi_1(BG, *) \to G$. It is universal in the sense that one has $h_{\boldsymbol{\alpha}}^* \boldsymbol{\iota} = \boldsymbol{\alpha}$ for a map $h_{\boldsymbol{\alpha}} : X \to BG$ corresponding to a gauge field $\boldsymbol{\alpha}$ on X. Furthermore the twisted cohomology groups $H^\bullet(BG, K, \boldsymbol{\iota})$ are defined for any G-module K. They are also called the cohomology groups of the group G and can be constructed in a purely algebraic manner, see [57]. The universal character of the field $\boldsymbol{\iota}$ implies that pullback through $h_{\boldsymbol{\alpha}}$ maps $H^\bullet(BG, K, \boldsymbol{\iota})$ to $H^\bullet(X, K, \boldsymbol{\alpha})$.

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C.2 Classifying spaces of crossed modules

Here we will describe classifying spaces of crossed modules. For our purposes, the following definition is suitable: $B\mathbb{G}$ is a connected CW-complex which contains $B\mathcal{E}$, the classifying space of the group $\mathcal{E}$, as a subcomplex and has homotopy groups

$$\pi_n(B\mathbb{G};*) = \begin{cases} \mathrm{coker}(\Delta) & \text{for } n = 1, \\ \ker(\Delta) & \text{for } n = 2, \\ 0 & \text{for } n \geq 3. \end{cases} \tag{C.1}$$

Furthermore, it is required that $\Pi_2(B\mathbb{G}, B\mathcal{E};*) \cong \mathbb{G}$. Again, we may assume that $B\mathbb{G}$ has exactly one 0-cell $*$, which we choose as the base point.

It is known that such space $B\mathbb{G}$ exists and is determined uniquely up to a homotopy equivalence by the above properties [54, 58–60]. The latter fact is also obtained as a simple corollary from the discussion in the appendix C.4, while the former is proven in the appendix C.5.

The property of $B\mathbb{G}$ most important for us is that field configurations $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ on X with flat $\boldsymbol{\varphi}$, modulo vertex and edge transformations, are in one-to-one correspondence with homotopy classes of maps $X \to B\mathbb{G}$. Clearly the flatness constraint may be lifted by considering maps $X_2 \to B\mathbb{G}$ instead. We remark also that homotopy classes of maps of triples $(X, X_1, X_0) \to (B\mathbb{G}, B\mathcal{E}, *)$ correspond to field configurations with flat $\boldsymbol{\varphi}$. Again, the flatness condition may be removed by replacing X with X_2. Finally, it will be clear from the proof that a map $(X, X_1, X_0) \to (B\mathbb{G}, B\mathcal{E}, *)$ is homotopic as a map of triples to one with image in $B\mathcal{E}$ if and only if the corresponding configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ has trivial $\boldsymbol{\varphi}$, i.e. $\varphi_f = 1$ for every face f.

For the purpose of the proof, we may again asssume that all maps $X \to B\mathbb{G}$ take X_0 to a base point $*$. Let us consider first homotopy classes of maps of X_1 into $B\mathbb{G}$ and $B\mathcal{E}$. Proceeding as in the above exposition of classifying spaces of groups one may show that they are in one-to-one correspondence with gauge equivalence classes of $\pi_1(B\mathbb{G}, *) \cong \mathrm{coker}(\Delta)$ and $\pi_1(B\mathcal{E}, *) \cong \mathcal{E}$-valued gauge fields on X_1, respectively. Furthermore, the map $[X_1, B\mathcal{E}] \to [X_1, B\mathbb{G}]$ induced by the inclusion of $B\mathcal{E}$ in $B\mathbb{G}$ corresponds to reduction modulo $\mathrm{im}(\Delta)$, so in particular it is surjective. Using the homotopy extension property of the pair (X, X_1) we conclude that any map $X \to B\mathbb{G}$ is homotopic to one which maps X_1 to $B\mathcal{E}$ and X_0 to $*$. Such map sends every edge e of X to a loop in $B\mathcal{E}$ based at $*$, and hence determines an element $\epsilon_e \in \pi_1(B\mathcal{E}, *) \cong \mathcal{E}$.

Now consider the problem of extending a map $h_{\boldsymbol{\epsilon}} : X_1 \to BG$ which determines an $\mathcal{E}$-valued gauge field $\boldsymbol{\epsilon}$ to X_2. For every face f we have to extend the map on the boundary whose homotopy class is given by the element $\bar{\epsilon}_{\partial f} \in \pi_1(B\mathbb{G}, *)$. An extension exists if and only if $\bar{\epsilon}_{\partial f} = 1$, i.e. if $\epsilon_{\partial f}$ belongs to $\mathrm{im}(\Delta)$. Homotopy class of this extension, regarded as a map of triples $(I^2, \partial I, *) \to (B\mathbb{G}, B\mathcal{E}, *)$, determines and is determined by an element $\varphi_f \in \pi_2(B\mathbb{G}, B\mathcal{E}, *)$ such that $\Delta\varphi_f = \epsilon_{\partial f}$. Summarizing, every homomorphism $\Pi_2(X_2, X_1; X_0) \to \mathbb{G}$ is realized by some map of triples $(X_2, X_1, X_0) \to (B\mathbb{G}, B\mathcal{E}, *)$. Conversely, any homotopy class of maps of triples is determined by the corresponding homomorphism. Thus a bijection $[(X_2, X_1, X_0), (B\mathbb{G}, B\mathcal{E}, *)] \cong \mathrm{Hom}(\Pi_2(X_2, X_1; X_0), \mathbb{G})$ is established.

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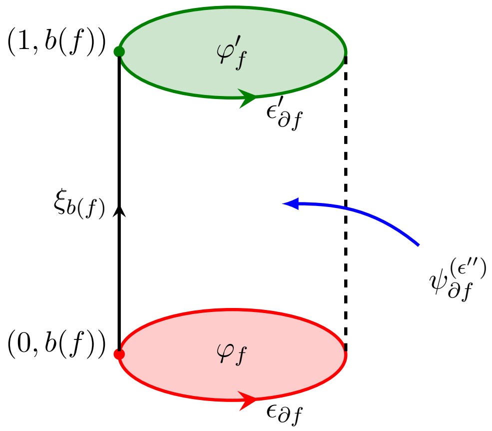

Figure 21. The cylinder is given a cellular structure with two 0-cells, indicated by bold dots. Three edges, indicated by solid lines, are mapped according to elements $\epsilon_{\partial f}, \epsilon'_{\partial f}, \xi_{b(f)} \in \mathcal{E}$. Two faces are given by bases of the cylinder and are mapped according to elements $\varphi_f, \varphi'_f \in \Phi$. The last face, based at $(1, b(f))$, forms the lateral surface. It is mapped according to the element $\psi_{\partial f}^{(\epsilon'')}$.

Next, let us take two maps $h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}, h_{\boldsymbol{\epsilon}',\boldsymbol{\varphi}'} : (X_2, X_1, X_0) \to (B\mathbb{G}, B\mathcal{E}, *)$, labeled by the corresponding field configurations, and consider the problem of deciding if they are homotopic as maps $X_2 \to B\mathbb{G}$. Thus we ask if the map $\{0,1\} \times X_2 \to B\mathbb{G}$ given by $h_{\boldsymbol{\epsilon}',\boldsymbol{\varphi}'}$ and $h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}$ on $\{1\} \times X_2$ and $\{0\} \times X_2$ extends to $I \times X_2$. By using the homotopy extension property of the pair $\Big(I \times X_2, (\{0,1\} \times X_2) \cup (I \times X_0)\Big)$ we conclude that every such extension is homotopic to one which sends $I \times X_0$ to $B\mathcal{E}$. Then cells $I \times \{v\}$ are sent to loops in $B\mathcal{E}$ described by elements $\xi_v \in \pi_1(B\mathcal{E}, *)$, which can be chosen at will. Next we extend through 2-cells. We encounter a problem analogous to the one illustrated on figure 20. An extension exists if $\epsilon'_e \left(\xi_{t(e)}\, \epsilon_e\, \xi_{s(e)}^{-1}\right)^{-1} \in \mathcal{E}$ represents a trivial element of $\pi_1(B\mathbb{G}, *) = \mathrm{coker}(\Delta)$. Assuming this is true, homotopy classes of extensions are described by elements $\psi_e \in \pi_2(B\mathbb{G}, B\mathcal{E}, *) = \Phi$ such that

$$\epsilon'_e = \Delta\psi_e\, \xi_{t(e)}\, \epsilon_e\, \xi_{s(e)}^{-1}. \tag{C.2}$$

From now we focus on one extension and proceed to extending through 3-cells. These are products of I and faces of X_2. A calculation shows that for every face f one encounters the problem of extending a map from the boundary of a cylinder, illustrated on figure 21, to its interior. This is possible if and only if the corresponding element $\varphi_f'^{-1} \left(\psi_{\partial f}^{(\epsilon'')}\, (\xi_{b(f)} \triangleright \varphi_f)\right) \in \pi_2(B\mathbb{G}, *) = \ker(\Delta)$ is trivial. Here $\boldsymbol{\epsilon}''$ is given by $\epsilon''_e = \xi_{t(e)}\, \epsilon_e\, \xi_{s(e)}^{-1}$.

Summarizing, a homotopy between $h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}$ and $h_{\boldsymbol{\epsilon}',\boldsymbol{\varphi}'}$ exists if and only if there exist collections $\boldsymbol{\xi} \in \mathcal{E}_X^{(0)}$ and $\boldsymbol{\psi} \in \Phi_X^{(1)}$ fitting in a diagram of the form presented on the figure 22. In other words, configurations $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ and $(\boldsymbol{\epsilon}', \boldsymbol{\varphi}')$ have to be related by the action of vertex and edge transformations.

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$$
\begin{tikzcd}
(\boldsymbol{\epsilon}, \boldsymbol{\varphi}) \arrow[rr] \arrow[dr, "\xi"] & & (\boldsymbol{\epsilon}', \boldsymbol{\varphi}') \\
& (\boldsymbol{\epsilon}'', \boldsymbol{\varphi}'') \arrow[ur, "\psi"] &
\end{tikzcd}
$$

Figure 22. Field configuration $(\boldsymbol{\epsilon}', \boldsymbol{\varphi}')$ is obtained from $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ by a vertex transformation $\boldsymbol{\xi}$ followed by an edge transformation $\boldsymbol{\psi}$.

We have completed the classification of homotopy classes of maps $X_2 \to B\mathbb{G}$. Now let us observe that restriction from X to X_2 defines maps

$$[X, B\mathbb{G}] \to [X_2, B\mathbb{G}], \qquad [(X, X_1, X_0), (B\mathbb{G}, B\mathcal{E}, *)] \to [(X_2, X_1, X_0), (B\mathbb{G}, B\mathcal{E}, *)].$$

We claim that they are injective. Indeed, suppose that $l, l' : X \to B\mathbb{G}$ are such that their restrictions to X_2 are homotopic. Every map $(\{0,1\} \times X) \cup (I \times X_2) \to B\mathbb{G}$ extends to $I \times X$, since $\pi_n(B\mathbb{G}; *)$ is trivial for $n \geq 3$. If the initial homotopy was a homotopy of maps of triples, then so is the extension. This completes the proof of the claim.

Next we ask when does a map $h : X_2 \to B\mathbb{G}$ extend to X. Firstly, the answer depends only on the homotopy class of h, by the homotopy extension property of (X, X_2). Thus we may assume that h is a map of triples $(X_2, X_1, X_0) \to (B\mathbb{G}, B\mathcal{E}, *)$ and write $h = h_{\boldsymbol{\epsilon}, \boldsymbol{\varphi}}$. Secondly, if an extension to X_3 exists, then there exists also an extension to X, by triviality of higher homotopy groups of $B\mathbb{G}$. It remains to decide when it is possible to extend through 3-cells. We consider a ball q. Its boundary is mapped to $B\mathbb{G}$ with homotopy class $\varphi_{\partial q} \in \pi_2(B\mathbb{G}; *) = \ker(\Delta)$. Thus an extension to whole X_3 exists if and only if $\boldsymbol{\varphi}$ is flat. Hence the proof that homotopy classes of maps $X \to B\mathbb{G}$ are in one-to-one correspondence with configurations with flat $\boldsymbol{\varphi}$ modulo vertex and edge transformations, as well as of the corresponding statement for the maps of triples, is completed.

C.3 Postnikov class

In this appendix we consider the following question: for which $\boldsymbol{\epsilon}$ there exists a flat $\boldsymbol{\varphi}$? First let us observe that the answer depends only on $\overline{\boldsymbol{\epsilon}}$ modulo gauge transformations since flatness of $\boldsymbol{\varphi}$ is invariant under vertex and edge transformations. Now let us choose one representative $\overline{\boldsymbol{\epsilon}}$, its lift to $\boldsymbol{\epsilon}$ and any $\boldsymbol{\varphi}$ satisfying the fake flatness condition. Next we define $\widehat{\delta}\boldsymbol{\varphi} : \pi_3(X_3, X_2; X_0) \to \ker(\Delta)$ by the formula $(\widehat{\delta}\varphi)_\omega = \varphi_{\partial\omega}$. Let us observe that it has the following properties:

- $\pi_1(X; X_0)$-equivariance, i.e. $(\widehat{\delta}\varphi)_{\gamma \triangleright \omega} = \overline{\epsilon}_\gamma \triangleright (\widehat{\delta}\varphi)_\omega$ for a path γ starting at the base point of ω. Thus $\widehat{\delta}\boldsymbol{\varphi}$ is a twisted 3-cochain, see appendix B.
- $\widehat{\delta}\boldsymbol{\varphi}$ is a twisted cocycle. The proof of this is analogous to the proof of the fact that δ^2 is trivial. Nevertheless, it is not necessarily true that $\widehat{\delta}\boldsymbol{\varphi}$ is in the image of δ: $\boldsymbol{\varphi}$, being valued in the non-abelian group Φ, is not a 2-cochain.
- The cohomology class of $\widehat{\delta}\boldsymbol{\varphi}$ does not depend on the choice of a lift of $\overline{\boldsymbol{\epsilon}}$ to $\boldsymbol{\epsilon}$ nor the choice of $\boldsymbol{\varphi}$. Indeed, edge transformations do not change $\widehat{\delta}\boldsymbol{\varphi}$ at all, while plaquette transformation $\boldsymbol{\chi}$ merely shifts it by $\delta\boldsymbol{\chi}$.

$$
\begin{tikzcd}
(X_2, X_1, X_0) \arrow[r, "h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}"] & (B\mathbb{G}, B\mathcal{E}, *) \\
(Y_2, Y_2, Y_0) \arrow[u, "l"] \arrow[ur, "h_{l^*\boldsymbol{\epsilon},l^*\boldsymbol{\varphi}}"'] &
\end{tikzcd}
$$

Figure 23. Pullback of field configurations may be defined in terms of the associated maps to the classifying space: $(l^*\boldsymbol{\epsilon}, l^*\boldsymbol{\varphi})$ corresponds to the map $l \circ h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}$.

$$
\begin{tikzcd}
X_2 \arrow[r, hook] \arrow[d, "h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}"'] & X \arrow[d, "h_{\overline{\boldsymbol{\epsilon}}}"] \\
B\mathbb{G} \arrow[r, "\Upsilon"] & B\mathrm{coker}(\Delta)
\end{tikzcd}
$$

Figure 24. Given a map $h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}} : X_2 \to B\mathbb{G}$ we may compose it with Υ and then extend to a map $X \to B\mathrm{coker}(\Delta)$, uniquely up to a homotopy. This extension corresponds to the gauge field $\overline{\boldsymbol{\epsilon}}$.

- $\widehat{\delta}\boldsymbol{\varphi}$ is trivial (i.e. $\varphi_{\partial\omega} = 1$ for every ω) if and only if $\boldsymbol{\varphi}$ is flat. Here we are using the fact that balls q generate $\pi_3(X_3, X_2; x)$ as a $\pi_1(X_1; x)$-module for any $x \in X_0$.

It is clear from the above properties that the cohomology class $[\widehat{\delta}\boldsymbol{\varphi}] \in H^3(X, \ker(\Delta), \overline{\boldsymbol{\epsilon}})$ is trivial if and only if $\overline{\boldsymbol{\epsilon}}$ is such that there exists a compatible configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ with flat $\boldsymbol{\varphi}$.

Cocyle $\widehat{\delta}\boldsymbol{\varphi}$ satisfies an important naturality property. Namely, let us consider a map of triples $l : (Y_2, Y_1, Y_0) \to (X_2, X_1, X_0)$ for some CW-complex Y. Then it makes sense to pull back a field configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ on X to a configuration $l^*(\boldsymbol{\epsilon}, \boldsymbol{\varphi}) = (l^*\boldsymbol{\epsilon}, l^*\boldsymbol{\varphi})$ on Y. One possible description of this pullback operation is through the diagram 23. Equivalently, $(l^*\boldsymbol{\epsilon}, l^*\boldsymbol{\varphi})$ is given by the composition

$$
\Pi_2(Y_2, Y_1; Y_0) \xrightarrow{l_*} \Pi_2(X_2, X_1; X_0) \xrightarrow{(\boldsymbol{\epsilon},\boldsymbol{\varphi})} \mathbb{G}.
$$

Clearly we have $l^*[\widehat{\delta}\boldsymbol{\varphi}] = [\widehat{\delta} l^*\boldsymbol{\varphi}] \in H^3(Y, \ker(\Delta), l^*\overline{\boldsymbol{\epsilon}})$. This innocuous-looking statement allows to relate $[\widehat{\delta}\boldsymbol{\varphi}]$ to a universal example.

The identity homomorphism $\pi_1(B\mathbb{G}, *) \to \mathrm{coker}(\Delta)$ determines, up to a homotopy, a map of pairs $\Upsilon : (B\mathbb{G}, *) \to (B\mathrm{coker}(\Delta), *)$. Therefore for a configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ on X (not necessarily with flat $\boldsymbol{\varphi}$) we have a commutative diagram of continuous maps presented on the figure 24. Suppose that there existed a map $\Xi : B\mathrm{coker}(\Delta) \to B\mathbb{G}$ such that $\Upsilon \circ \Xi$ was homotopy equivalent to the self-identity map on $B\mathrm{coker}(\Delta)$. Then the map $h = \Xi \circ h_{\overline{\boldsymbol{\epsilon}}} : X \to B\mathbb{G}$ would be such that $\Upsilon \circ h$ is homotopic to $h_{\overline{\boldsymbol{\epsilon}}}$, yielding a conclusion that some configuration $(\boldsymbol{\epsilon}', \boldsymbol{\varphi}')$ with $\overline{\boldsymbol{\epsilon}}' = \overline{\boldsymbol{\epsilon}}$ and flat $\boldsymbol{\varphi}'$ exists. This is not always true, so the desired Ξ does not always exist. On the other hand one could attempt to construct it cell-by-cell. The obstruction to do this is a universal example for the cohomology classes $[\widehat{\delta}\boldsymbol{\varphi}]$, as we will now demonstrate.

Let $\overline{\boldsymbol{\iota}}$ be the tautological $\mathrm{coker}(\Delta)$-valued gauge field on $B\mathrm{coker}(\Delta)$. We may construct its lift to a $\mathbb{G}$-valued field configuration $(\boldsymbol{\iota}, \boldsymbol{o})$ on $B\mathbb{G}$, which determines a mapping $h_{\boldsymbol{\iota},\boldsymbol{o}}$: $(B\mathrm{coker}(\Delta)_2, B\mathrm{coker}(\Delta)_1, *) \to (B\mathbb{G}, B\mathcal{E}, *)$. Thus we may form the cocycle $\widehat{\delta}\boldsymbol{o}$, which is

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a representative of the so-called Postnikov class

$$\beta = [\widehat{\delta}\boldsymbol{o}] \in H^3(B\text{coker}(\Delta), \ker(\Delta), \bar{\iota}). \tag{C.3}$$

We reiterate the fact that β does not depend on the choice of $\boldsymbol{\iota}$ and $\boldsymbol{o}$, although the representative cocycle $\widehat{\delta}\boldsymbol{o}$ certainly does. The map $h_{\boldsymbol{\iota},\boldsymbol{o}}$ induces the identity homomorphism between fundamental groups, and conversely, any map with this property is homotopic to one of the form $h_{\boldsymbol{\iota},\boldsymbol{o}}$ for some $(\boldsymbol{\iota},\boldsymbol{o})$. Thus if β is nontrivial, a right homotopy inverse Ξ of Υ does not exist. Conversely, if β is trivial, then some $h_{\boldsymbol{\iota},\boldsymbol{o}}$ extends to the whole $B\text{coker}(\Delta)$. Denoting the extension by Ξ, we have that $\Upsilon \circ \Xi$ induces the identity map on $\pi_1(B\text{coker}(\Delta), *)$ and hence is homotopic to the identity map, by the classification of maps valued in classifying spaces of groups.

We claim that for any field configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ on X one has the relation

$$[\widehat{\delta}\boldsymbol{\varphi}] = h^*_{\bar{\boldsymbol{\epsilon}}}\beta. \tag{C.4}$$

Indeed, consider the field configuration $(\boldsymbol{\epsilon}', \boldsymbol{\varphi}') = (h^*_{\bar{\boldsymbol{\epsilon}}}\boldsymbol{\iota}, h^*_{\bar{\boldsymbol{\epsilon}}}\boldsymbol{o})$. Then $\bar{\boldsymbol{\epsilon}}' = \bar{\boldsymbol{\epsilon}}$, which implies that $\widehat{\delta}\boldsymbol{\varphi}' = h^*_{\bar{\boldsymbol{\epsilon}}}\widehat{\delta}\boldsymbol{o}$ and $\widehat{\delta}\boldsymbol{\varphi}$ are cohomologous. In particular, a configuration $(\boldsymbol{\epsilon}'', \boldsymbol{\varphi}'')$ with flat $\boldsymbol{\varphi}''$ and $\bar{\boldsymbol{\epsilon}}'' = \bar{\boldsymbol{\epsilon}}$ exists if and only if the pullback $h^*_{\bar{\boldsymbol{\epsilon}}}\beta$ of the Postnikov class is trivial.

C.4 Homomorphisms and weak equivalences

In this appendix we will assume that the 1-skeleton of $B\mathbb{G}$ is contained in $B\mathcal{E}$. This is possible, because the inclusion of $B\mathcal{E}$ in $B\mathbb{G}$ induces an epimorphism of fundamental groups, see [55, p. 219]. With this condition the identity map on $B\mathbb{G}$ may be regarded as a map of triples $(B\mathbb{G}, B\mathbb{G}_1, *) \to (B\mathbb{G}, B\mathcal{E}, *)$. Thus it determines a $\mathbb{G}$-valued field configuration $(\boldsymbol{\kappa}, \boldsymbol{\eta})$ on $B\mathbb{G}$, called the tautological configuration. This configuration has flat $\boldsymbol{\eta}$. The corresponding map takes $B\mathcal{E}$ to $B\mathcal{E}$, so $\boldsymbol{\eta}$ restricted to $B\mathcal{E}$ is trivial: $\eta_f = 1$ for every face f in $B\mathcal{E}$ (but not necessarily for faces in $B\mathbb{G}$).

Configuration $(\boldsymbol{\kappa}, \boldsymbol{\eta})$ is universal: if $h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}$ is a cellular map $(X_2, X_1, X_0) \to (B\mathbb{G}, B\mathcal{E}, *)$ corresponding to a configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$, then $h^*_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}(\boldsymbol{\kappa}, \boldsymbol{\eta}) = (\boldsymbol{\epsilon}, \boldsymbol{\varphi})$. This is because in this case the map l on the figure 23 is simply the inclusion of $(B\mathbb{G}_2, \mathbb{G}_1, *)$ in $(B\mathbb{G}, B\mathcal{E}, *)$, so $h_{l^*\boldsymbol{\epsilon}, l^*\boldsymbol{\varphi}} = h_{\boldsymbol{\epsilon},\boldsymbol{\varphi}}$.

Now let $\mathbb{G}'$ be another crossed module and let $(E, F) : \mathbb{G} \to \mathbb{G}'$ be a homomorphism. Then $(E(\boldsymbol{\kappa}), F(\boldsymbol{\eta}))$ is a $\mathbb{G}'$-valued configuration on $B\mathbb{G}$, so it determines a map $(B\mathbb{G}_2, B\mathbb{G}_1, *) \to (B\mathbb{G}', B\mathcal{E}', *)$, unique up to a homotopy of maps of triples. Since $\boldsymbol{\eta}$ was flat, so is $F(\boldsymbol{\eta})$. Thus the corresponding map extends to whole $B\mathbb{G}$, uniquely up to a homotopy of maps of triples $(B\mathbb{G}, B\mathbb{G}_1, *) \to (B\mathbb{G}', B\mathcal{E}', *)$. We denote the extension by $B(E, F)$. Furthermore, $F(\boldsymbol{\eta})$ is trivial on $B\mathbb{G}$, so (perhaps after a homotopy of maps of triples) $B(E, F)$ takes $B\mathcal{E}$ to $B\mathcal{E}'$. Then $B(E, F)$ induces a homomorphism

$$(B(E, F))_* : \mathbb{G} = \Pi_2(B\mathbb{G}, B\mathcal{E}; *) \to \Pi_2(B\mathbb{G}', B\mathcal{E}'; *) = \mathbb{G}'.$$

We claim that $(B(E, F))_* = (E, F)$. Indeed, let $i : (B\mathbb{G}_2, B\mathbb{G}_1, *) \to (B\mathbb{G}, B\mathcal{E}, *)$ be the inclusion. Since $B(E, F)$ corresponds to the configuration $(E(\boldsymbol{\kappa}), F(\boldsymbol{\eta}))$, we have that the

$$
\begin{tikzcd}
\mathbb{G} \arrow[dr, bend left, "(E,F)"] & \\
\Pi_2(B\mathbb{G}_2, B\mathbb{G}_1; *) \arrow[u, "(\boldsymbol{\kappa},\boldsymbol{\eta})"] \arrow[r, "(E(\boldsymbol{\kappa}),F(\boldsymbol{\eta}))"] \arrow[d, "i_*"] & \mathbb{G}' \\
\mathbb{G} \arrow[ur, bend right, "(B(E,F))_*"'] &
\end{tikzcd}
$$

Figure 25. The upper triangle commutes by definition of the configuration $(E(\boldsymbol{\kappa}), F(\boldsymbol{\eta}))$, while the lower triangle commutes by construction of the map $B(E,F)$.

$$
\begin{tikzcd}
0 & \pi_1(B\mathbb{G}_2;*) \arrow[l] \arrow[d, "\cong"] & \pi_1(B\mathbb{G}_1;*) \arrow[l] \arrow[d, two heads] & \pi_2(B\mathbb{G}_2, B\mathbb{G}_1;*) \arrow[l] \arrow[d] & \pi_2(B\mathbb{G}_2;*) \arrow[l] \arrow[d, two heads] & 0 \arrow[l] \arrow[d, "\cong"] \\
0 & \pi_1(B\mathbb{G};*) \arrow[l] & \pi_1(B\mathcal{E};*) \arrow[l] & \pi_2(B\mathbb{G}, B\mathcal{E};*) \arrow[l] & \pi_2(B\mathbb{G};*) \arrow[l] & 0 \arrow[l]
\end{tikzcd}
$$

Figure 26. The upper row and the lower row are pieces of long exact sequences of homotopy groups for pointed pairs $(B\mathbb{G}_2, B\mathbb{G}_1)$ and $(B\mathbb{G}, B\mathcal{E})$, respectively. Downwards arrows are induced by the inclusion map.

diagram of homomorphisms of crossed modules on figure 25 is commutative. On the other hand we have $i_* = (\boldsymbol{\kappa}, \boldsymbol{\eta})$, by construction of $(\boldsymbol{\kappa}, \boldsymbol{\eta})$. Therefore

$$(B(E,F))_* \circ i_* = (E,F) \circ i_*. \tag{C.5}$$

Maps $\pi_1(B\mathbb{G}_1;*) \to \pi_1(B\mathbb{G};*)$ and $\pi_2(B\mathbb{G}_2, B\mathbb{G}_1;*) \to \pi_2(B\mathbb{G}, B\mathcal{E};*)$ are epimorphisms, so (C.5) implies the validity of the claim. Indeed, surjectivity of the first homomorphism is clear. Secondly, we know that the inclusion $B\mathbb{G}_2 \to B\mathbb{G}$ induces an epimorphism of second homotopy groups and that the second homotopy groups of $B\mathbb{G}_1$ and $B\mathcal{E}$ are trivial. Hence by naturality of the long exact sequence of relative homotopy groups, the diagram on figure 26 is commutative with exact rows. The proof is completed by the four lemma.

We have proven that $(B(E,F))_* = (E,F)$, so in particular the maps of first and second homotopy groups induced by $B(E,F)$ are $\overline{E}$ and $\overline{F}$, respectively. Thus if (E,F) is a weak isomorphism, then $B(E,F)$ is a homotopy equivalence, by Whitehead's theorem [50, p. 346]. Thus it induces a bijection $[X, B\mathbb{G}] \cong [X, B\mathbb{G}']$ for every space X, so topological gauge theories based on $\mathbb{G}$ and $\mathbb{G}'$ are equivalent. More explicitly, this equivalence is given by mapping a $\mathbb{G}$-valued configuration $(\boldsymbol{\epsilon}, \boldsymbol{\varphi})$ on X to a $\mathbb{G}'$-valued configuration $(E(\boldsymbol{\epsilon}), F(\boldsymbol{\varphi}))$.

We remark that the above result implies that $B\mathbb{G}$ is determined uniquely up to a homotopy equivalence, a fact which we have never used. Indeed, if $\widetilde{B\mathbb{G}}$ is another construction of the classifying space of $\mathbb{G}$, the above construction gives a homotopy equivalence $B\mathbb{G} \to \widetilde{B\mathbb{G}}$ induced by the identity homomorphism $\mathbb{G} \to \mathbb{G}$.

$$\begin{tikzcd}
0 & \pi_1(B\mathbb{G}_{2\frac{1}{2}};*) \arrow[l] \arrow[d, equal] & \pi_1(B\mathcal{E}_3;*) \arrow[l] \arrow[d, equal] & \pi_2(B\mathbb{G}_{2\frac{1}{2}},B\mathcal{E}_3;*) \arrow[l, "\partial"'] \arrow[d, two heads, "p"] & \pi_2(B\mathbb{G}_{2\frac{1}{2}};*) \arrow[l] \arrow[d, two heads, "\overline{p}"] & \pi_2(B\mathcal{E}_3;*) \arrow[l] \arrow[d, equal] \\
& \operatorname{coker}(\Delta) & \mathcal{E} \arrow[l] & \Phi \arrow[l, "\Delta"'] & \ker(\Delta) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Figure 27. Commutative diagram whose upper row is a portion of the long exact sequence of homotopy groups of the pair $(B\mathbb{G}_{2\frac{1}{2}}, B\mathcal{E}_3)$.

C.5 Construction of classifying spaces

In this appendix we fix a crossed module $\mathbb{G}$ and construct a classifying space $B\mathbb{G}$ together with its subcomplex $B\mathcal{E}$ by gluing cells. In the process we will repeatedly use standard results [55, p. 215] concerning the effect of attaching cells on homotopy groups, in particular the fact that the n-th homotopy group is not changed by attaching cells of dimension greater than $n+1$ (say, by the cellular approximation theorem). The latter is true also for relative homotopy groups.

Firstly, for the 0-skeleton we take a single point $*$. To proceed further, we choose a presentation of $\mathcal{E}$, i.e. a set $\{\epsilon_i\}_{i\in I}$ and relations $\{\rho_j\}_{j\in J}$. For each $i \in I$ we attach to $*$ a single edge, so that $B\mathbb{G}_1 = B\mathcal{E}_1$ is $\bigvee_{i\in I} S^1$, a bouquet of circles. Now the fundamental group of $B\mathcal{E}_1$ is free with generators indexed by the set I. We denote the generator corresponding to $i \in I$ by ϵ_i. Each relation ρ_j is a word in the alphabet $\{\epsilon_i\}_{i\in I}$, so it defines an element of the fundamental group of $B\mathcal{E}_1$. Space $B\mathcal{E}_2$ is formed by attaching to $B\mathcal{E}_1$ a 2-cell for each $j \in J$, with an attaching map of homotopy class $\rho_j \in \pi_1(B\mathcal{E}_1;*)$. Then $\pi_1(B\mathcal{E}_2;*) = \mathcal{E}$.

Next we choose a set $\{\varphi_k\}_{k\in K} \subseteq \Phi$ such that the elements $\epsilon \rhd \varphi_k$ with any $\epsilon \in \mathcal{E}$ generate the group Φ. Space $B\mathbb{G}_2$ is formed from $B\mathcal{E}_2$ by attaching a 2-cell for each $k \in K$, with attaching maps of homotopy classes $\Delta\varphi_k \in \mathcal{E} = \pi_1(B\mathcal{E}_2;*)$. Then the fundamental group of $B\mathbb{G}_2$ is $\operatorname{coker}(\Delta)$.

Space $B\mathcal{E}_3$ is formed by attaching 3-cells to $B\mathcal{E}_2$ in such a way that $\pi_2(B\mathcal{E}_3;*)$ becomes trivial, e.g. one 3-cell for each element of a set of generators of $\pi_2(B\mathcal{E}_3;*)$. Then an auxillary space $B\mathbb{G}_{2\frac{1}{2}}$ is formed by attaching to $B\mathbb{G}_2$ the 3-cells of $B\mathcal{E}$, or equivalently [55, p. 49] by gluing to $B\mathcal{E}_3$ those 2-cells of $B\mathbb{G}$ which are not in $B\mathcal{E}$.

To proceed further, we need to understand the group $\widetilde{\Phi} := \pi_2(B\mathbb{G}_{2\frac{1}{2}}, B\mathcal{E}_3;*)$. Since $B\mathbb{G}_{2\frac{1}{2}}$ is obtained from $B\mathcal{E}_3$ by attaching faces, Whitehead's theorem applies and we have that $\widetilde{\Phi}$ is generated by elements $\epsilon \rhd \phi_k$ (with ϕ_k - the generator corresponding to the k-th face), subject only to relations following from Peiffer identities in the crossed module $\widetilde{\mathbb{G}} := \Pi_2(B\mathbb{G}_{2\frac{1}{2}}, B\mathcal{E}_3;*)$. Furthermore, the boundary homomorphism $\pi_2(B\mathbb{G}_{2\frac{1}{2}}, B\mathcal{E}_3;*) \to \pi_1(B\mathcal{E}_3;*) = \mathcal{E}$ is given by

$$\partial\,(\epsilon \rhd \phi_k) \mapsto \epsilon\,\Delta\varphi_k\,\epsilon^{-1}. \tag{C.6}$$

The assignment $p(\epsilon \rhd \phi_k) = \epsilon \rhd \varphi_k$ defines a group epimorphism $p : \widetilde{\Phi} \to \Phi$. Furthermore, (id, p) is a homomorphism of crossed modules $\widetilde{\mathbb{G}} \to \mathbb{G}$. All that is summarized by the commutative diagram with exact rows presented on figure 27. We let $\{\lambda_l\}_{l\in L}$ be a set of generators of $\ker(p)$. Then for each l we have that $\partial\lambda_l = \Delta(p(\lambda_l))$ is trivial. On the other

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hand the kernel of ∂ may be identified with $\pi_2(B\mathbb{G}_{2\frac{1}{2}};*)$, since $\pi_2(B\mathcal{E}_3;*)$ is trivial. Thus we may regard λ_l as elements of $\pi_2(B\mathbb{G}_{2\frac{1}{2}};*)$. They generate the kernel of $\overline{p}$. The space $B\mathbb{G}_3$ is formed from $B\mathbb{G}_{2\frac{1}{2}}$ by attaching 3-cells with attaching maps of homotopy classes λ_l. Then we have $\Pi_2(B\mathbb{G}_3, B\mathcal{E}_3;*) = \mathbb{G}$.

At this point all homotopy groups up to degree 2 are as desired. The procedure may be continued inductively: for every $k \geq 4$ space $B\mathcal{E}_k$ is obtained from $B\mathcal{E}_{k-1}$ by attaching k-cells in such a way that $\pi_{k-1}(B\mathcal{E}_k;*)$ becomes trivial. Then $B\mathbb{G}_k$ is formed from $B\mathbb{G}_{k-1}$ by attaching all k-cells of $B\mathcal{E}$ and possible some additional cells needed to assure that $\pi_{k-1}(B\mathbb{G}_k;*)$ becomes trivial. Finally, we let $B\mathbb{G}$ (resp. $B\mathcal{E}$) be the union of all $B\mathbb{G}_k$ (resp. $B\mathcal{E}_k$), endowed with the weak topology.

References

[1] L.D. Landau, *On the theory of phase transitions*, *Zh. Eksp. Teor. Fiz.* **7** (1937) 19 [*Phys. Z. Sowjetunion* **11** (1937) 26] [*Ukr. J. Phys.* **53** (2008) 25] [INSPIRE].

[2] J.M. Kosterlitz and D.J. Thouless, *Ordering, metastability and phase transitions in two-dimensional systems*, *J. Phys. C* **6** (1973) 1181 [INSPIRE].

[3] X.G. Wen, *Topological order in rigid states*, *Int. J. Mod. Phys. B* **4** (1990) 239 [INSPIRE].

[4] X. Chen, Z.C. Gu and X.G. Wen, *Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order*, *Phys. Rev. B* **82** (2010) 155138 [arXiv:1004.3835] [INSPIRE].

[5] C.L. Kane and E.J. Mele, *Z_2 topological order and the quantum spin Hall effect*, *Phys. Rev. Lett.* **95** (2005) 146802 [cond-mat/0506581] [INSPIRE].

[6] L. Fu, C. Kane and E. Mele, *Topological insulators in three dimensions*, *Phys. Rev. Lett.* **98** (2007) 106803 [cond-mat/0607699] [INSPIRE].

[7] T.H. Hansson, V. Oganesyan and S.L. Sondhi, *Superconductors are topologically ordered*, *Annals Phys.* **313** (2004) 497 [cond-mat/0404327] [INSPIRE].

[8] E. Fradkin, *Field theories of condensed matter physics*, Cambridge University Press, Cambridge, U.K. (2013).

[9] C.-K. Chiu, J.C. Teo, A.P. Schnyder and S. Ryu, *Classification of topological quantum matter with symmetries*, *Rev. Mod. Phys.* **88** (2016) 035005.

[10] T.O. Wehling, A.M. Black-Schaffer and A.V. Balatsky, *Dirac materials*, *Adv. Phys.* **63** (2014) 1 [arXiv:1405.5774] [INSPIRE].

[11] A.Y. Kitaev, *Fault-tolerant quantum computation by anyons*, *Annals Phys.* **303** (2003) 2 [quant-ph/9707021] [INSPIRE].

[12] Z. Wang, *Topological quantum computation*, Amer. Math. Soc., U.S.A. (2010).

[13] M. Atiyah, *Topological quantum field theories*, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1989) 175 [INSPIRE].

[14] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989) 351 [INSPIRE].

[15] J.C. Baez, *Four-dimensional BF theory as a topological quantum field theory*, Lett. Math. Phys. **38** (1996) 129.

[16] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Commun. Math. Phys. **129** (1990) 393 [INSPIRE].

[17] V.G. Turaev, *Quantum invariants of knots and* 3*-manifolds*, Walter de Gruyter & Co., Berlin, Germany (1994).

[18] L. Crane and D. Yetter, *A categorical construction of* 4d *topological quantum field theories*, Quant. Topol. **3** (1993) 120.

[19] M.A. Levin and X.-G. Wen, *String net condensation: a physical mechanism for topological phases*, Phys. Rev. B **71** (2005) 045110 [cond-mat/0404617] [INSPIRE].

[20] K. Walker and Z. Wang, $(3+1)$*-TQFTs and topological insulators*, Front. Phys. **7** (2011) 150.

[21] D.N. Yetter, *TQFT's from homotopy* 2*-types*, J. Knot Theor. Ramif. **02** (1993) 113.

[22] F.J. Wegner, *Duality in generalized Ising models and phase transitions without local order parameters*, J. Math. Phys. **12** (1971) 2259 [INSPIRE].

[23] K.G. Wilson, *Confinement of quarks*, Phys. Rev. D **10** (1974) 2445 [INSPIRE].

[24] J.B. Kogut, *An introduction to lattice gauge theory and spin systems*, Rev. Mod. Phys. **51** (1979) 659 [INSPIRE].

[25] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and duality*, JHEP **04** (2014) 001 [arXiv:1401.0740] [INSPIRE].

[26] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized global symmetries*, JHEP **02** (2015) 172 [arXiv:1412.5148] [INSPIRE].

[27] K. Holland and U.-J. Wiese, *The center symmetry and its spontaneous breakdown at high temperatures*, in *At the frontier of particle physics. Handbook of QCD*, volume 3, World Scientific, Singapore (2001), pg. 1909.

[28] F. Benini, C. Córdova and P.-S. Hsin, *On* 2*-group global symmetries and their anomalies*, JHEP **03** (2019) 118 [arXiv:1803.09336] [INSPIRE].

[29] A. Kapustin and R. Thorngren, *Higher symmetry and gapped phases of gauge theories*, Prog. Math. **324** (2017) 177.

[30] X. Chen, Z.-C. Gu, Z.-X. Liu and X.-G. Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, Phys. Rev. B **87** (2013) 155114 [arXiv:1106.4772] [INSPIRE].

[31] M. Kalb and P. Ramond, *Classical direct interstring action*, Phys. Rev. D **9** (1974) 2273 [INSPIRE].

[32] M. Henneaux and C. Teitelboim, *p-form electrodynamics*, Found. Phys. **16** (1986) 593 [INSPIRE].

[33] J.C. Baez, *Higher Yang-Mills theory*, hep-th/0206130 [INSPIRE].

[34] H. Pfeiffer, *Higher gauge theory and a non-Abelian generalization of* 2*-form electrodynamics*, Annals Phys. **308** (2003) 447 [hep-th/0304074] [INSPIRE].

[35] J.C. Baez and J. Huerta, *An invitation to higher gauge theory*, Gen. Rel. Grav. **43** (2011) 2335 [arXiv:1003.4485] [INSPIRE].

[36] T. Porter, *Topological quantum field theories from homotopy n-types*, J. Lond. Math. Soc. **58** (1998) 723.

[37] J.F. Martins and T. Porter, *On Yetter's invariant and an extension of the Dijkgraaf-Witten invariant to categorical groups*, Theor. Appl. Categories **18** (2007) 118 [math.QA/0608484].

[38] F. Girelli, H. Pfeiffer and E.M. Popescu, *Topological higher gauge theory: from BF to BFCG theory*, J. Math. Phys. **49** (2008) 032503 [arXiv:0708.3051] [INSPIRE].

[39] D.J. Williamson and Z. Wang, *Hamiltonian models for topological phases of matter in three spatial dimensions*, Annals Phys. **377** (2017) 311 [arXiv:1606.07144] [INSPIRE].

[40] A. Bullivant, M. Calçada, Z. Kádár, P. Martin and J.F. Martins, *Topological phases from higher gauge symmetry in* $3 + 1$ *dimensions*, Phys. Rev. B **95** (2017) 155118 [arXiv:1606.06639] [INSPIRE].

[41] C. Delcamp and A. Tiwari, *From gauge to higher gauge models of topological phases*, JHEP **10** (2018) 049 [arXiv:1802.10104] [INSPIRE].

[42] A. Bullivant, M. Calçada, Z. Kádár, J.F. Martins and P. Martin, *Higher lattices, discrete two-dimensional holonomy and topological phases in* $(3 + 1)D$ *with higher gauge symmetry*, Rev. Math. Phys. **32** (2019) 2050011.

[43] S. Gukov and A. Kapustin, *Topological quantum field theory, nonlocal operators, and gapped phases of gauge theories*, arXiv:1307.4793 [INSPIRE].

[44] A. Kapustin and R. Thorngren, *Topological field theory on a lattice, discrete theta-angles and confinement*, Adv. Theor. Math. Phys. **18** (2014) 1233 [arXiv:1308.2926] [INSPIRE].

[45] M. Mackaay and R. Picken, *Holonomy and parallel transport for Abelian gerbes*, Adv. Math. **170** (2002) 287.

[46] J. Baez and U. Schreiber, *Higher gauge theory: 2-connections on 2-bundles*, hep-th/0412325 [INSPIRE].

[47] U. Schreiber and K. Waldorf, *Parallel transport and functors*, *J. Homotopy Relat. Struct.* **4** (2009) 187 [arXiv:0705.0452].

[48] J.C. Baez and A.D. Lauda, *Higher dimensional algebra. V: 2-groups*, Theor. Appl. Categ. **12** (2004) 423 [math.QA/0307200].

[49] J.P. Ang and A. Prakash, *Higher categorical groups and the classification of topological defects and textures*, arXiv:1810.12965 [INSPIRE].

[50] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, U.K. (2002).

[51] J.H.C. Whitehead, *Combinatorial homotopy. II*, Bull. Amer. Math. Soc. **55** (1949) 453.

[52] R. Brown, *On the second relative homotopy group of an adjunction space: an exposition of a theorem of J.H.C. Whitehead*, J. Lond. Math. Soc. **s2-22** (1980) 146.

[53] A. Terras, *Fourier analysis on finite groups and applications*, Cambridge University Press, Cambridge, U.K. (1999).

[54] S. MacLane and J.H.C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950) 41.

[55] G.W. Whitehead, *Elements of homotopy theory*, Springer, New York, NY, U.S.A. (1978).

JHEP03(2021)282

[56] D. Husemöller, *Fibre bundles*, Springer, New York, NY, U.S.A. (1994).

[57] K.S. Brown, *Cohomology of groups*, Springer, New York, NY, U.S.A. (1982).

[58] J.-L. Loday, *Spaces with finitely many non-trivial homotopy groups*, *J. Pure Appl. Alg.* **24** (1982) 179.

[59] R. Brown and P.J. Higgins, *The classifying space of a crossed complex*, *Math. Proc. Camb. Phil. Soc.* **110** (1991) 95.

[60] R. Brown, P.J. Higgins and R. Sivera, *Nonabelian algebraic topology*, *EMS Tracts Math.* **15** (2011) 1.

Published for SISSA by Springer
Received: May 23, 2021
Revised: August 16, 2021
Accepted: August 17, 2021
Published: September 10, 2021

Dynamics of a lattice 2-group gauge theory model

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Abstract: We study a simple lattice model with local symmetry, whose construction is based on a crossed module of finite groups. Its dynamical degrees of freedom are associated both to links and faces of a four-dimensional lattice. In special limits the discussed model reduces to certain known topological quantum field theories. In this work we focus on its dynamics, which we study both analytically and using Monte Carlo simulations. We prove a factorization theorem which reduces computation of correlation functions of local observables to known, simpler models. This, combined with standard Krammers-Wannier type dualities, allows us to propose a detailed phase diagram, which form is then confirmed in numerical simulations. We describe also topological charges present in the model, its symmetries and symmetry breaking patterns. The corresponding order parameters are the Polyakov loop and its generalization, which we call a Polyakov surface. The latter is particularly interesting, as it is beyond the scope of the factorization theorem. As shown by the numerical results, expectation value of Polyakov surface may serve to detects all phase transitions and is sensitive to a value of the topological charge.

Keywords: Lattice Quantum Field Theory, Higher Spin Symmetry, Spontaneous Symmetry Breaking, Topological Field Theories

ArXiv ePrint: 2105.05671

https://doi.org/10.1007/JHEP09(2021)068

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1 Introduction

Higher gauge theories are physical models which generalize conventional gauge theory by associating degrees of freedom to geometric objects of dimension higher than one. Perhaps the best known example is the p-form electrodynamics [1], whose discretized version can be naturally formulated in terms of degrees of freedom associated to p-cells, e.g. plaquettes for $p = 2$. These degrees of freedom are subject to redundancy described by group valued functions on the set of $(p-1)$-cells. In the case of $p = 1$ this reduces to degrees of freedom on links with gauge transformations given by arbitrary functions defined on lattice sites.

Already in [1] it was argued that gauge theories with $p \geq 2$ are necessarily abelian, essentially because there exist no well-behaved orderings on surfaces. There is a way to bypass this argument, inspired by higher category theory [2–4]. For p not exceeding 2, it is typically formulated in terms of 2-groups [5] or, equivalently, crossed modules [6]. Surface observables in 2-group gauge theories are still valued in an abelian group, but in general they are computed in terms of genuinely non-abelian local degrees of freedom associated to links and plaquettes.

There exists also a concept of (global) higher form symmetries [7], whose relation with higher gauge theories is similar to the relation between ordinary symmetries and gauge theories. Examples of models admitting higher symmetries have been known for a long time, and among gauge theories they are in fact the rule rather than an exception. Nevertheless, systematic study of higher symmetries seems to have begun relatively recently.

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Higher gauge theories have been proposed [8, 9] as effective field theories describing vacua of conventional gauge theories. They also provide interesting examples [10–15] of Topological Quantum Field Theories (TQFTs) [16, 17], and hence are expected to describe certain gapped topological phases of many body quantum systems. In [10] the existence of Symmetry Protected Topological (SPT) phases protected by higher symmetries was proposed. Another motivation to study higher gauge theories is provided by its relation with bosonization in arbitrary dimension [18–20]. Furthermore, certain models in string theory may be described as higher gauge theories [21].

Yetter's model [11] is a TQFT based on a crossed module of finite groups. Its hamiltonian realizations resembling the Kitaev's toric code were constructed in [22–24]. In [25] a common generalization of the Yetter's model, 2-form $\mathbb{Z}_n$ electrodynamics and lattice Yang-Mills theory has been proposed. It is a genuinely dynamical model, formulated in the hamiltonian formalism, which reduces to a TQFT only in certain limits. In this work we consider an analogous model formulated in terms of state sums (discrete functional integrals). We focus on one relatively simple crossed module, but some of our results are true in general. In order to make the paper more accessible, we have decided to define everything explicitly using notations standard in lattice gauge theory. We refer to [25] for an exposition of the slightly more involved formalism of crossed modules and proofs of various algebraic facts used in the present text.

Full definition of the model under consideration is given in subsection 2.1. Its extended observables are discussed in subsection 2.2. We identify topological charges and higher symmetries: 1-form symmetry $\mathbb{Z}_2^{(1)}$ and 2-form symmetry $\mathbb{Z}_2^{(2)}$. We discuss the theoretical possibility of symmetry breaking and provide suitable order parameters. In subsection 2.3 we show that computation of a large class of observables, including all local observables, may be reduced to calculation of averages in simpler models: 1-form $\mathbb{Z}_2$ gauge theory and 2-form $\mathbb{Z}_2$ gauge theory. This includes the statement that plaquette observables (constructed from link degrees of freedom in the usual way) are uncorrelated with cube observables (constructed from plaquette degrees of freedom), which is not obvious from the form of the action. This factorization theorem is not valid for the surface observable which is the order parameter of the $\mathbb{Z}_2^{(2)}$ symmetry. In subsection 2.4 we use the factorization theorem and Kramers-Wannier type dualities to formulate a proposal for the phase diagram. We describe critical points, symmetry breaking patterns and renormalization group fixed points governing the infrared physics.

Section 3 is devoted to Monte Carlo study of the proposed model in dimension $D = 4$. Simulation algorithm is described in subsection 3.1. Since the general method is fairly standard, we discuss in detail only those aspects that are specific to the case at hand. In subsection 3.2 we present numerical results for expectation values of local observables. These results confirm the phase structure obtained from duality arguments. The most interesting, in our view, results of simulations are presented in subsection 3.3. They concern behaviour of order parameters for higher symmetries $\mathbb{Z}_2^{(1)}$ and $\mathbb{Z}_2^{(2)}$. It is found that order parameters for the latter not only exhibit sharp dependence on both coupling constants, but are also sensitive to the topological charge.

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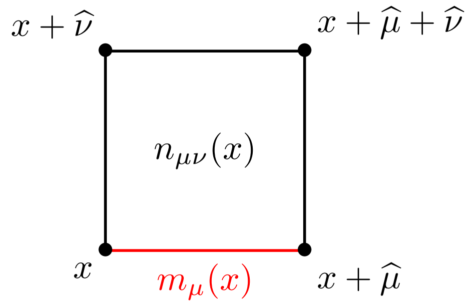

Figure 1. Independent degrees of freedom are associated to links $m_\mu(x)$ and faces $n_{\mu\nu}(x)$. The latter should not be confused with plaquette observables $f_{\mu\nu}(x)$ constructed from links.

The paper contains two appendices. In the appendix A, we discuss the construction of non-spherical surface observables using the general language of crossed modules. Appendix B contains a brief discussion of models with continuous gauge fields similar to the one studied here. It is argued that there are two terms in the action that have to be included in order to obtain a natural generalization of Yang-Mills theory. Analogy with these two terms is among our main motivations to focus on the particular form of the action chosen in this paper. Such lattice action can be constructed for any crossed module of finite groups. We emphasize that our main analytic result, the factorization theorem, is valid for every crossed module with such choice of an action. Nevertheless, for a different form of the action its conclusion may not hold.

2 Description of the model

2.1 Degrees of freedom, action and gauge freedom

Coordinates of a lattice site form a tuple $x = (x_1, \ldots, x_D)$ with integer x_μ. Unit vector in the direction $\mu \in \{1, \ldots, D\}$ will be denoted by $\widehat{\mu}$. We choose periodic boundary conditions, i.e. x_μ is identified with $x_\mu + L_\mu$, where L_μ is the extent of the system in the μ-th direction.

Addition and multiplication in $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is always performed modulo four. We consider a model with degrees of freedom of two types, both valued in $\mathbb{Z}_4$ (see figure 1):

- $m_\mu(x)$, associated with the link between x and $x + \widehat{\mu}$,
- $n_{\mu\nu}(x)$, associated with the square with corners x, $x+\widehat{\mu}$, $x+\widehat{\mu}+\widehat{\nu}, x+\widehat{\nu}$ (called face).

They are subject to a constraint (for every x and $\mu < \nu$) called *fake flatness*:

$$2n_{\mu\nu}(x) = m_\mu(x) + m_\nu(x + \widehat{\mu}) - m_\mu(x + \widehat{\nu}) - m_\nu(x). \tag{2.1}$$

The right hand side is a plaquette built of m variables as in the ordinary lattice gauge theory. It is convenient to denote it by $f_{\mu\nu}(x)$. We note that $n_{\mu\nu}(x)$ is determined by the link variables only modulo two and that $f_{\mu\nu}(x)$ has to be even (but $m_\mu(x)$ not necessarily so).

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Out of elementary degrees of freedom one may construct observables associated to cubes:

$$\begin{aligned} g_{\mu\nu\rho}(x) = & -n_{\mu\nu}(x) + n_{\mu\rho}(x) - n_{\nu\rho}(x) \\ & + (-1)^{m_\rho(x)} n_{\mu\nu}(x+\widehat{\rho}) - (-1)^{m_\nu(x)} n_{\mu\rho}(x+\widehat{\nu}) + (-1)^{m_\mu(x)} n_{\nu\rho}(x+\widehat{\mu}). \end{aligned} \tag{2.2}$$

The six terms in this formula correspond to six faces of a cube. It can be shown that fake flatness enforces all $g_{\mu\nu\rho}(x)$ to be even.

Observable $f_{\mu\nu}(x)$ is the Wilson line along the boundary of an elementary rectangle. In the present model it is possible to construct also higher dimensional analogues of Wilson lines, which could be called Wilson surfaces. Observable $g_{\mu\nu\rho}(x)$ is the Wilson surface along an elementary cube.

We choose the following action:

$$S = -J_1 \sum_x \sum_{\mu<\nu} (-1)^{\frac{f_{\mu\nu}(x)}{2}} - J_2 \sum_x \sum_{\mu<\nu<\rho} (-1)^{\frac{g_{\mu\nu\rho}(x)}{2}} = J_1 S_1(m) + J_2 S_2(m,n), \tag{2.3}$$

with $J_1, J_2 \geq 0$. The first term is the Wilson action for m variables. It is minimized if all plaquettes $f_{\mu\nu}(x)$ are equal to zero. Every plaquette equal to 2 costs $2J_1$ units of action. The second term is a higher dimensional analogue of the Wilson term for the n variables. Again, it is minimized if all cubes $g_{\mu\nu\rho}(x)$ are equal to zero. Every excited cube costs $2J_2$ units of action.

Degrees of freedom superficially seem to interact with each other, since they are related by the fake flatness condition and since $g_{\mu\nu\rho}(x)$ (which enters the action directly) depends on both degrees of freedom. However, as it will be demonstrated later, this interaction does not affect local dynamics, i.e. plaquettes are uncorrelated with cubes and furthermore correlation functions of plaquettes and cubes depend only on J_1 and only on J_2, respectively. On the other hand, the impact of the interaction can be seen in averages of nonlocal order parameters. Numerical evidence supporting this statement is presented in section 3.

The fake flatness constraint (2.1) and the action (2.3) are invariant under gauge transformations of two types. Gauge transformations associated to sites are parametrized by elements $\xi(x) \in \mathbb{Z}_4$. They act according to the formulas:

$$m_\mu(x) \mapsto m_\mu(x) + \xi(x+\widehat{\mu}) - \xi(x), \tag{2.4a}$$

$$n_{\mu\nu}(x) \mapsto (-1)^{\xi(x)} n_{\mu\nu}(x). \tag{2.4b}$$

Gauge transformations associated to links are parametrized by $\psi_\mu(x) \in \{0,2\} \subsetneq \mathbb{Z}_4$ and act as

$$m_\mu(x) \mapsto m_\mu(x), \tag{2.5a}$$

$$n_{\mu\nu}(x) \mapsto n_{\mu\nu}(x) + \psi_\mu(x) + \psi_\nu(x+\widehat{\mu}) - \psi_\mu(x+\widehat{\nu}) - \psi_\nu(x). \tag{2.5b}$$

Only gauge invariant quantities will be regarded as observables. In this work we consider $f_{\mu\nu}(x)$, $g_{\mu\nu\rho}(x)$ and order parameters described in subsection 2.2.

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2.2 Nonlocal order parameters and symmetries

Polyakov loop, a particular Wilson line winding around one of the directions of the lattice, is defined by the formula

$$p_\mu(x) = \exp\left(\frac{\mathrm{i}\pi}{2}\sum_{j=0}^{L_\mu-1} m_\mu(x+j\widehat{\mu})\right). \tag{2.6}$$

Its possible values are ± 1 and $\pm\mathrm{i}$, in contrast to plaquette observables which take only two possible values. For configurations with $f_{\mu\nu}(x)=0$ for all x, value of $p_\mu(x)$ is independent of x. This is not true for general configurations. On the other hand, quantity Q_μ defined by

$$Q_\mu = p_\mu(x)^2 \tag{2.7}$$

is independent of x, which follows from the fact that all $f_{\mu\nu}(x)$ are even. We will call it the topological charge. Each Q_μ may take two possible values, 1 or -1, so the whole set of field configurations decomposes into 2^D disjoint sectors. We note that local constraint-preserving transformations in the set of all field configurations cannot change the topological charge, since that requires changing $p_\mu(x)$ for all x.

For every μ there exists a symmetry of the action which leaves all plaquettes, cubes and $\{p_\nu(x)\}_{\nu\neq\mu}$ unchanged, but flips the sign of $p_\mu(x)$ (and hence preserves Q_μ). It is given by

$$m_\nu(x) \mapsto m_\nu(x) + 2\delta_{\mu,\nu}\delta_{x_\mu,0}, \tag{2.8a}$$
$$n_{\nu\rho}(x) \mapsto n_{\nu\rho}(x). \tag{2.8b}$$

We will call it the electric 1-form symmetry. As a consequence of this symmetry the expectation value of $p_\mu(x)$ vanishes.

We are unaware of a symmetry which changes $p_\mu(x)$ by a factor of i (and hence flips the sign of Q_μ). Nevertheless, it will turn out to be useful to consider the Q_μ-reversing transformation

$$m_\nu(x) \mapsto m_\nu(x) + \delta_{\mu,\nu}\delta_{x_\mu,0}, \tag{2.9a}$$
$$n_{\nu\rho}(x) \mapsto n_{\nu\rho}(x). \tag{2.9b}$$

It preserves fake flatness and all $f_{\mu\nu}(x)$, so it is a symmetry of S_1. However, it changes values of cube observables, so it is not a symmetry of the full action.

We would like to address the question whether the symmetry (2.8) can be spontaneously broken. We insist on gauge invariance and locality of the action, so it is not possible to include a symmetry breaking term in the action. On the other hand, in a putative phase with unbroken symmetry the infinite volume limit of the volume average of $p_\mu(x)$ in a fixed typical field configuration is expected to vanish. This happens for small J_1, because then plaquette observables fluctuate strongly, so the signs of $p_\mu(x)$ and $p_\mu(y)$ are essentially independent if the transverse distance $|x-y|_\perp = \sqrt{\sum_{\nu\neq\mu}(x_\nu-y_\nu)^2}$ is large. More precisely, $p_\mu(x)p_\mu(y)^{-1}$ may be understood as a Wilson loop bounding area $L_\mu|x-y|_\perp$, so its expectation value is expected to decay exponentially with $|x-y|_\perp$.

To quantify the above discussion, we consider

$$P_\mu = \left| V_\perp^{-1} \sum_x p_\mu(x) \right| , \tag{2.10}$$

with the sum taken over x in a plane transverse to the μ-th direction. Here $V_\perp = \prod_{\nu\neq\mu} L_\nu$ is the transverse volume. Squaring this definition we find

$$P_\mu^2 = V_\perp^{-2} \sum_{x,y} p_\mu(x) p_\mu(y)^{-1}. \tag{2.11}$$

There are $V_\perp^2$ terms in this sum, each of which has modulus one. After taking expectation value, only $O(V_\perp)$ terms, with $|x-y|_\perp$ comparable to the correlation length survive. Therefore the average of P_μ^2 decreases as $V_\perp^{-1}$, so P_μ decreases as $V_\perp^{-\frac{1}{2}}$:

$$P_\mu \sim V_\perp^{-\frac{1}{2}}, \qquad L_\mu \text{ fixed, } V_\perp \to \infty. \tag{2.12}$$

By spontaneous breaking of the symmetry (2.8) we shall understand violation of this scaling law, so that P_μ remains nonzero in the limit of infinite transverse volume:

$$P_\mu \sim \text{const} \neq 0, \qquad L_\mu \text{ fixed, } V_\perp \to \infty. \tag{2.13}$$

Note that it may still be true that $P_\mu \to 0$ as $L_\mu \to \infty$.

There exists a surface observable analogous to the Polyakov loop. It may be thought of as a Wilson surface winding around two lattice directions. Its construction is slightly more involved. We choose a plane through a fixed site x parallel to directions $\mu < \nu$. Morally speaking, we would like to add $n_{\mu\nu}(y)$ with y running through all sites in the chosen plane. However, this does not give a gauge invariant quantity. To fix this, we have to choose for every y a path from y to x (which we take to be entirely contained in the chosen plane) and weigh $n_{\mu\nu}(y)$ by a parallel transport factor $\prod(-1)^{m_\rho(z)}$, where the product is taken over all links forming the chosen path. Then the sum, denoted $\Sigma_{\mu\nu}(x)$, is gauge-invariant and even. This is discussed in the broader context of crossed modules in the appendix A. We define

$$p_{\mu\nu}(x) = \exp\left(\frac{\mathrm{i}\pi}{2}\Sigma_{\mu\nu}(x)\right), \tag{2.14}$$

which will be called the Polyakov plane.

We remark that our notation is fully justified only if either $Q_\mu = Q_\nu = 1$ or all $f_{\mu\nu}(x)$ vanish, because otherwise $p_{\mu\nu}(x)$ depends on the choice of paths needed to construct it. This hints at the possibility that expectation values of $p_{\mu\nu}(x)$ may depend both on the two coupling constants and on the topological charge. This will be corroborated by results in section 3.

There exists a symmetry which flips the sign of $p_{\mu\nu}(x)$:

$$m_\rho(x) \mapsto m_\rho(x), \tag{2.15a}$$
$$n_{\rho\sigma}(x) \mapsto n_{\rho\sigma}(x) + 2\delta_{\mu,\rho}\delta_{\nu,\sigma}\delta_{x_\mu,0}\delta_{x_\nu,0}, \qquad \rho < \sigma. \tag{2.15b}$$

We will call it electric 2-form symmetry. It implies that the expectation value of $p_{\mu\nu}(x)$ vanishes.

By analogy with the Polyakov loop, we consider the quantity

$$P_{\mu\nu} = \left| V_{\perp}^{-1} \sum_{x} p_{\mu\nu}(x) \right| , \tag{2.16}$$

where $V_{\perp} = \prod_{\rho \neq \mu,\nu} L_{\rho}$ and the sum is taken over a plane transverse to $\widehat{\mu}$ and $\widehat{\nu}$. We will say that the symmetry (2.15) is broken if $P_{\mu\nu}$ has nonzero limit as $V_{\perp} \to \infty$.

2.3 Reduction of dynamics to simpler models

In this subsection we will show how to express certain averages with respect to the action (2.3) in terms of averages in simpler models. We will make use of constraint-preserving moves in the space of field configurations. Firstly, the link moves:

$$m_{\mu}(x) \mapsto m_{\mu}(x) + 2\psi_{\mu}(x), \tag{2.17a}$$

$$\begin{aligned} n_{\mu\nu}(x) \mapsto n_{\mu\nu}(x) &+ (-1)^{m_{\mu}(x)}\psi_{\mu}(x) + (-1)^{m_{\mu}(x+\widehat{\nu})+m_{\nu}(x)}\psi_{\nu}(x+\widehat{\mu}) \\ &- (-1)^{m_{\mu}(x+\widehat{\nu})+m_{\nu}(x)}\psi_{\mu}(x+\widehat{\nu}) - (-1)^{m_{\nu}(x)}\psi_{\nu}(x), \end{aligned} \tag{2.17b}$$

with arbitrary $\psi_{\mu}(x) \in \mathbb{Z}_4$. They reduce to gauge transformations (2.5) if $\psi_{\mu}(x)$ is even. In general they change the value of plaquette observables $f_{\mu\nu}(x)$, but not of the cube observables $g_{\mu\nu\rho}(x)$. Secondly, the face moves:

$$m_{\mu}(x) \mapsto m_{\mu}(x), \tag{2.18a}$$

$$n_{\mu\nu}(x) \mapsto n_{\mu\nu}(x) + \chi_{\mu\nu}(x), \tag{2.18b}$$

with $\chi_{\mu\nu}(x) \in \{0, 2\} \subsetneq \mathbb{Z}_4$. They preserve $f_{\mu\nu}(x)$, but change values of $g_{\mu\nu\rho}(x)$.

Moves described above generate a group. Every move may be represented as a sequence of local moves with only one nonzero $\psi_{\mu}(x)$ or $\chi_{\mu\nu}(x)$. Since $m_{\mu}(x)$ are always either unchanged or shifted by an even amount, topological charges Q_{μ} are invariant.

We claim that any two configurations with equal topological charges can be related by a sequence of local moves and a gauge transformation. Indeed, first consider two configurations with equal $m_{\mu}(x)$. Then, by fake flatness, all $n_{\mu\nu}(x)$ differ by even numbers, so the two configurations are related by a face transformation. This reduces the proof of the claim to showing that $m_{\mu}(x)$ can be made equal by a sequence of link moves and a gauge transformation. The only gauge invariant functions of $m_{\mu}(x)$ are Wilson lines, which can be taken along contractible loops or non-contractible loops. The former are expressible in terms of $f_{\mu\nu}(x)$ and have to be even. The latter are also even on the account of the assumption about topological charges, since every loop can be built of contractible loops and Polyakov loops. This proves that up to pure gauge terms, the difference of $m_{\mu}(x)$ is even. Thus they are related by a transformation of the form (2.17a).

The average of an observable O is $\langle O \rangle = \frac{Z_O(J_1,J_2)}{Z(J_1,J_2)}$, where $Z(J_1, J_2) = Z_{\mathbf{1}}(J_1, J_2)$ and

$$Z_O(J_1, J_2) = \sum_{m,n} f(Q) e^{-J_1 S_1(m) - J_2 S_2(m,n)} O(m, n), \tag{2.19}$$

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in which the sum over m, n is restricted by the constraint. Function $f(Q)$ is a weight given to the sector with topological charge Q. The simplest choice is $f(Q) = 1$, while restriction to $Q = Q'$ with fixed Q' corresponds to $f(Q) = \delta_{Q,Q'}$. We consider an observable of the form $O = O_1 O_2$ such that:

- O_1 can by expressed solely in terms of plaquette observables $f_{\mu\nu}(x)$ (thus it can be P_μ),
- O_2 is invariant with respect to gauge transformations and link moves, e.g. it is an arbitrary function of cube observables $g_{\mu\nu\rho}(x)$.

We define the quantity

$$W_{O_2}(J_2; m) = \sum_n e^{-J_2 S_2(m,n)} O_2(m,n). \tag{2.20}$$

It is invariant with respect to gauge transformations and link moves of m variables, so it depends on m only through Q_μ. Therefore we write $W_{O_2}(J_2; m) = W_{O_2,Q}(J_2)$, which gives

$$Z_{O_1 O_2}(J_1, J_2) = \sum_m f(Q) e^{-J_1 S_1(m)} O_1(m) W_{O_2,Q}(J_2). \tag{2.21}$$

We divide the summation over m into topological sectors. The sum over m with fixed Q will be denoted by index $m|Q$:

$$Z_{O_1 O_2}(J_1, J_2) = \sum_Q f(Q) W_{O_2,Q}(J_2) \sum_{m|Q} e^{-J_1 S_1(m)} O_1(m). \tag{2.22}$$

Sum $\sum\limits_{m|Q} e^{-J_1 S_1(m)} O_1(m)$ does not depend on Q by symmetry (2.9) of S_1. Finally:

$$Z_{O_1 O_2}(J_1, J_2) = \left(\sum_{m|1} e^{-J_1 S_1(m)} O_1(m) \right) \left(\sum_Q f(Q) W_{O_2,Q}(J_2) \right). \tag{2.23}$$

In the remaining sum over m we have configurations of m variables such that every Wilson loops is even. Such configuration is gauge equivalent to one with all $m_\mu(x)$ even. Furthermore, every gauge orbit has 4^{N_1-1} representatives (in which N_1 is the number of links), out of which 2^{N_1-1} is such that all $m_\mu(x)$ are even. Therefore we may restrict the sum over m to configurations with even $m_\mu(x)$ at the small cost of including a factor 2^{N_1-1}. Then the sum over m gives the Wegner model [26], so

$$Z_{O_1 O_2}(J_1, J_2) = 2^{N_1-1} Z_{O_1}^{\text{Wegner}}(J_1) \sum_Q f(Q) W_{O_2,Q}(J_2). \tag{2.24}$$

This gives

$$\langle O_1 O_2 \rangle = \frac{Z_{O_1 O_2}(J_1, J_2)}{Z(J_1, J_2)} = \frac{Z_{O_1}^{\text{Wegner}}(J_1)}{Z^{\text{Wegner}}(J_1)} \cdot \frac{\sum\limits_Q f(Q) W_{O_2,Q}(J_2)}{\sum\limits_Q f(Q) W_{\mathbf{1},Q}(J_2)}, \tag{2.25}$$

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from which we can draw the following conclusions:

- factorization $\langle O_1 O_2 \rangle = \langle O_1 \rangle \langle O_2 \rangle$,
- $\langle O_1 \rangle$ does not depend on J_2 and weights f, and is equal to the average in Wegner's model,
- $\langle O_2 \rangle$ does not depend on J_1.

This factorization theorem is the main result of this section. We would like to remark that its derivation remains valid also for models based on general crossed modules of finite groups, as long as the action is a sum of a term depending only on plaquette observables and a term depending only on cube observables. This observation follows from the fact that the presented proof relies only on general properties of gauge transformations and constraint-preserving moves. These were discussed in detail in [25].

Next we turn to the question on how $\langle O_2 \rangle$ depends on the topological charge sector. We will argue that for thermodynamic quantities dependence becomes negligible in the infinite volume limit. This will be confirmed already for quite small lattices by results of simulations presented in section 3.

Consider, for concreteness, the case $Q_1 = -1$, $Q_\mu = 1$ for $\mu \neq 1$. Such choice of topological charge may be realized by the gauge field

$$m_\mu(x) = \delta_{\mu,1} \delta_{x_\mu,0}. \tag{2.26}$$

It is supported on a plane, so switching it on (without modifying $n_{\mu\nu}(x)$ variables) may change the value of at most $\binom{D-1}{2} \prod\limits_{\mu \neq 1} L_\mu$ cubes. Hence we have

$$|S_2(m,n) - S_2(0,n)| \leq 2 \binom{D-1}{2} \prod_{\mu \neq 1} L_\mu. \tag{2.27}$$

It follows that $\frac{W_{\mathbf{1},Q}(J_2)}{W_{\mathbf{1},\mathrm{trivial}}(J_2)} = \frac{\sum\limits_n e^{-J_2 S_2(0,n)} e^{-J_2 (S_2(m,n) - S_2(0,n))}}{\sum\limits_n e^{-J_2 S_2(0,n)}}$ obeys an estimate

$$e^{-4J_2 \binom{D-1}{2} \prod_{\mu \neq 1} L_\mu} \leq \frac{W_{\mathbf{1},Q}(J_2)}{W_{\mathbf{1},\mathrm{trivial}}(J_2)} \leq e^{4J_2 \binom{D-1}{2} \prod_{\mu \neq 1} L_\mu}. \tag{2.28}$$

Taking logarithms gives an estimate on the difference of free energies per unit volume:

$$\left| \frac{1}{J_2} \log(W_{\mathbf{1},Q}(J_2)) - \frac{1}{J_2} \log(W_{\mathbf{1},\mathrm{trivial}}(J_2)) \right| \leq 4 \binom{D-1}{2} \prod_{\mu \neq 1} L_\mu. \tag{2.29}$$

We recall that the free energy is an extensive quantity. On the other hand, the right hand side divided by the volume decays as L_1^{-1} as $L_1 \to \infty$. We conclude that in the infinite volume limit, the free energies per unit volume become equal in all topological sectors.

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2.4 Phase diagram proposal for $D = 4$

In this subsection we restrict attention to dimension $D = 4$, although some parts of the discussion are valid also for other dimensions.

In the case $D = 4$, Wegner's model has a single phase transition [26], which is of first order. Its exact position

$$J_1^{\text{crit}} = \frac{1}{2}\text{arsinh}(1) \approx 0.441 \tag{2.30}$$

may be calculated using Kramers-Wannier type self-duality[1] [27]. There exist two renormalization group fixed points at $J_1 = 0$ and $J_1 = \infty$. Two phases may be interpreted as their basins of attraction.

At the point $J_1 = 0$, degrees of freedom become completely random and hence the theory is trivial. Effect of a small, but nonzero J_1 may be calculated using the strong coupling expansion [28]. One finds that Wilson loops obey the area law, and hence the electric 1-form symmetry is unbroken.

At $J_1 = \infty$ the system is constrained to configurations which minimize the action. Thus all plaquette observables vanish and Polyakov loops become independent of position. Up to gauge transformations, minima of the action are labeled by values of Polyakov loops. In Wegner's model each P_μ takes 2 possible values, so there exist 16 minima. They all have the same value of the action, because they are connected by the electric 1-form symmetry. However, in order for the system to get from one minimum to another using local moves only, it has to overcome an infinite action barrier. Even for finite J_1 (but large, so that a typical configuration is close to a minimum) the height of the barrier is of order $J_1 V_\perp$, so one may expect the electric 1-form symmetry to be broken.

The link variable sector of our model is slightly different in that the Polyakov loop takes four, rather than two possible values. However, it becomes essentially equivalent to the Wegner's model after restricting to a single topological charge sector.

Next we turn to the local dynamics of plaquette degrees of freedom. There exists a Kramers-Wannier duality between $W_{\mathbf{1},\text{trivial}}(J_2)$ and the Ising model partition function[2] with

$$\sinh(2J_2)\sinh(2J_{\text{Ising}}) = 1. \tag{2.31}$$

In the Ising model one expects a single continuous phase transition[3] whose position reported in [30] is $J_{\text{Ising}}^{\text{crit}} = 0.149647(5)$. This corresponds to a continuous phase transition in our model at

$$J_2^{\text{crit}} = 0.953294(1). \tag{2.32}$$

The critical point of the Ising model is expected to be described by a massless scalar field theory. It admits one relevant perturbation, given by the mass term. Therefore the fixed

[1] Strictly speaking, the duality relates the partition function of the Wegner's model to the partition function summed over flat background 2-form $\mathbb{Z}_2$ gauge fields. However, these gauge fields have negligible effect on thermodynamic quantities, which can be shown analogously as in the last paragraph of subsection 2.3.

[2] Again, this is exact only if the Ising model partition function is summed over background $\mathbb{Z}_2$ gauge fields.

[3] We remark that in [29] a weakly first order phase transition was suggested.

point at $J_2 = J_2^{\rm crit}$ is repulsive. The only other fixed points at $J_2 = 0$ and $J_2 = \infty$ describe physics in phases $J_2 < J_2^{\rm crit}$ and $J_2 > J_2^{\rm crit}$, respectively.

Quite analogously to the Wegner's model, the electric 2-form symmetry is unbroken in the small J_2 phase. The situation is much more interesting for large J_2. To gain some orientation about this case, we consider the limit $J_2 = \infty$, in which configurations are constrained to minimize S_2. As shown in the appendix A, Polyakov surfaces $p_{\mu\nu}(x)$ become independent of x if in addition either $J_1 = \infty$ (i.e. for configurations minimizing also S_1) or if topological charges are trivial. Therefore we expect that the 2-form symmetry is broken if $J_2 > J_2^{\rm crit}$ and $J_1 > J_1^{\rm crit}$. In the phase $J_2 > J_2^{\rm crit}$, $J_1 < J_1^{\rm crit}$ we can make this conclusion only for the sector with trivial topological charge. On the other hand, numerical results in section 3 show that in the sector with $Q_\mu = -1$ the symmetry is restored. We find this result quite striking.

The following picture emerges. Our model has four phases, each corresponding to one attractive renormalization group fixed point. In each of the fixed points local dynamics becomes trivial, but some nonlocal observables remain important:

- $(J_1, J_2) = (0, 0)$: $\mathbb{Z}_2$ topological charges Q_μ,
- $(J_1, J_2) = (\infty, 0)$: $\mathbb{Z}_4$ Polyakov loops P_μ,
- $(J_1, J_2) = (\infty, \infty)$: $\mathbb{Z}_4$ Polyakov loops P_μ and $\mathbb{Z}_2$ Polyakov surfaces $P_{\mu\nu}$, completely independent of each other,
- $(J_1, J_2) = (0, \infty)$: $\mathbb{Z}_4$ Polyakov loops P_μ and $\mathbb{Z}_2$ Polyakov surfaces $P_{\mu\nu}$, with an interplay between topological charges and Polyakov surfaces.

We remark that the four renormalization group fixed points described here may be identified with four integrable hamiltonians described in [25].

3 Monte Carlo study

3.1 Simulation method

In the numerical setup, we keep the extent of three directions equal $L_0 = L_1 = L_2 = L$, whereas L_3 will be varied separately. We denote the entire volume by $V = L^3 \times L_3$. We will also use the notation

$$(x_0, x_1, x_2, x_3) = (x, y, z, t). \tag{3.1}$$

For any observable we define its statistical expectation value, denoted by $\langle \cdot \rangle$, as the arithmetic mean over samples from a single Markov chain, and in some cases a weighted average of expectation values from multiple Markov chains. In most cases we perform a single simulation where we gather around 10^5 measurements, from which we estimate the average and its standard deviation, taking into account autocorrelations. We do the latter by explicitly calculating the autocorrelation function and integrating it up to the first non-positive element to quantify the autocorrelation time $\tau_{\rm int}$. In the following figures, all data points are shown together with their statistical uncertainties, which however may be

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smaller than the symbol size and hence invisible. In some cases we have performed up to four parallel simulations in order to increase the statistics and to check for ergodicity.

All simulations are performed using an intertwined application of Metropolis [31–33] and over-relaxation steps [34–38]. These are two independent update steps coming in pairs: one for updating the link variables and another to update face variables. We now describe both in more details.

Metropolis steps are based on local changes separately for both kind of degrees of freedom. We use (2.17a) to update the link variables and (2.18a) for the face variables. We remind that by construction such moves preserve the fake-flatness constraint. As a consequence, the move (2.17a) changes both link and face variables. If the constraint was satisfied by the initial configuration, it will be satisfied during the successive application of any of the above changes. Any two configurations can be linked by a finite-length chain of such local movements, which ensures that the simulations are ergodic. Each new configuration ν is obtained from a previous configuration μ by a local change of a randomly chosen degree of freedom and is subject to an accept/reject step with a probability given by

$$p_A(\mu \to \nu) = \min\left\{1, e^{S(\mu)-S(\nu)}\right\}. \tag{3.2}$$

Over-relaxation steps are made of non-local moves (2.8) and (2.15), which flip the signs of Polyakov lines P_μ and Polyakov planes $P_{\mu\nu}$, respectively. Since such transformations do not change the value of the action of a given configuration, they would be always accepted. Hence they are not subject to the accept/reject step. It is known that the incorporation of such moves between Metropolis moves reduces autocorrelation times significantly [34–38].

The local moves (2.17a) and (2.18a) cannot change the value of the topological charge. Hence, the simulation is limited to the topological sector given by the value of the topological charge of the initial configuration. In the following we discuss two independent chains of simulations, one performed in the trivial topological sector ($Q_\mu = 1$ for all μ) and the second performed in the sector with $Q_0 = -1$, see (2.7). We construct the latter by starting from an initial configuration where all the link and face variables are set to 0. Subsequently we set $m_0(0, y, z, t) = 1$ for all y, z, t, thus enforcing $P_0 = \mathrm{i}$ and hence $Q_0 = -1$.

The above algorithm with the accept/reject as in (3.2) satisfies the detailed balance condition, which together with the ergodicity of the local moves, guarantees the correctness of the entire algorithm in a given topological sector.

In order to identify the thermalization region of the Markov chain we usually perform an additional simulation with the same parameters, which we start from a so-called hot initial configuration. The latter is constructed by randomizing as much as possible all the degrees of freedom. To be more precise, we set $m_\mu(x, y, z, t)$ to 0 or 2 with equal probabilities, and subsequently adjust $n_{\mu\nu}(x, y, z, t)$ variables to satisfy the fake-flatness constraint. We do this by evaluating all plaquette variables $f_{\mu\nu}(x, y, z, t)$ and then setting $n_{\mu\nu} = \frac{1}{2} f_{\mu\nu}(x, y, z, t) + q$, where q is a random variable taking values 0 and 2 with equal probability. In both simulations, started from a cold and hot configuration, we monitor two local variables (3.3) and (3.4). Recording of relevant observables is started only when the two monitored quantities attain compatible values.

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3.2 Numerical results for local observables

In this section we discuss two local observables: plaquettes and cubes

$$F = \Big|\frac{1}{6V}\sum_{x,y,z,t}\sum_{\mu<\nu} f_{\mu\nu}(x,y,z,t)\Big|, \tag{3.3}$$

$$G = \Big|\frac{1}{4V}\sum_{x,y,z,t}\sum_{\mu<\nu<\rho} g_{\mu\nu\rho}(x,y,z,t)\Big|. \tag{3.4}$$

According to the factorization theorem (2.25), we expect that $\langle F\rangle$ does not depend on J_2 and $\langle G\rangle$ does not depend on J_1. Our first numerical results confirm these conclusions. In figure 2 we show the average values $\langle F\rangle$ and $\langle G\rangle$ as functions of J_1 and J_2 separately. Plots of data obtained in different topological sectors are also indistinguishable, up to statistical uncertainties.

In figure 2 we demonstrate the dependence of $\langle F\rangle$ and $\langle G\rangle$ on J_1 (two panels in the upper row) and J_2 (two panels in the lower row) coupling constants. Motivated by our expectations regarding the phase diagram of the system, i.e. existence of four distinct phases, as described in section 2.4, linked to the corners of the phase space given by $(J_1, J_2) = (0,0)$, $(0,\infty)$, $(\infty,0)$ and (∞,∞), we select values of J_1 and J_2 representing each phase:

$$J_1 = 0.43 \text{ or } 0.46, \tag{3.5}$$

$$J_2 = 0.10 \text{ or } 1.10. \tag{3.6}$$

When varying one of the coupling constants we keep the other in one of the two values.

We clearly see in figure 2 that $\langle G\rangle$ does not depend on J_1, i.e. the values are constant and compatible within their statistical uncertainties for the entire range of J_1 values investigated. Similarly, $\langle F\rangle$ does not depend on J_2. Near the location of the expected first order phase transition in J_1, value of $\langle F\rangle$ drops significantly. We demonstrate the nature of this phase transition in the left panel of figure 3. The panel reproduces the results from [39], where the hysteresis in the average plaquette action in the four-dimensional Wegner model [26] was interpreted as a clear sign of a first order phase transition.

As far as $\langle G\rangle$ is concerned, the lower right panel shows a rather smooth dependence. The part of the action proportional to J_2 is a function of $\langle G\rangle$, hence we conclude that also the action itself has a continuous dependence on J_2. This is in agreement with the expected nature of the phase transition in J_2 being second order. We corroborate this with the results shown in the right panel of figure 3, where fluctuations of $\langle G\rangle$ are shown to exhibit a drastic change around J_2^{crit}. To be precise, we plot $\sqrt{\langle (G-\langle G\rangle)^2\rangle V^{-1}}$. Again, all results for $\langle G\rangle$ show no dependence on the change in the J_1 coupling constant.

The results shown in figure 2 provide an illustration of the factorization theorem. Moreover, they support the expected existence of four phases at the four corners of the phase diagram. The more detailed results shown in figure 3 suggest that the location of the critical couplings where the phase transitions occur, agree within the accuracy of our simulations with the predictions (2.30) and (2.32). Hence, already the simple, local observables such as F and G provide valuable information about the system. We now turn

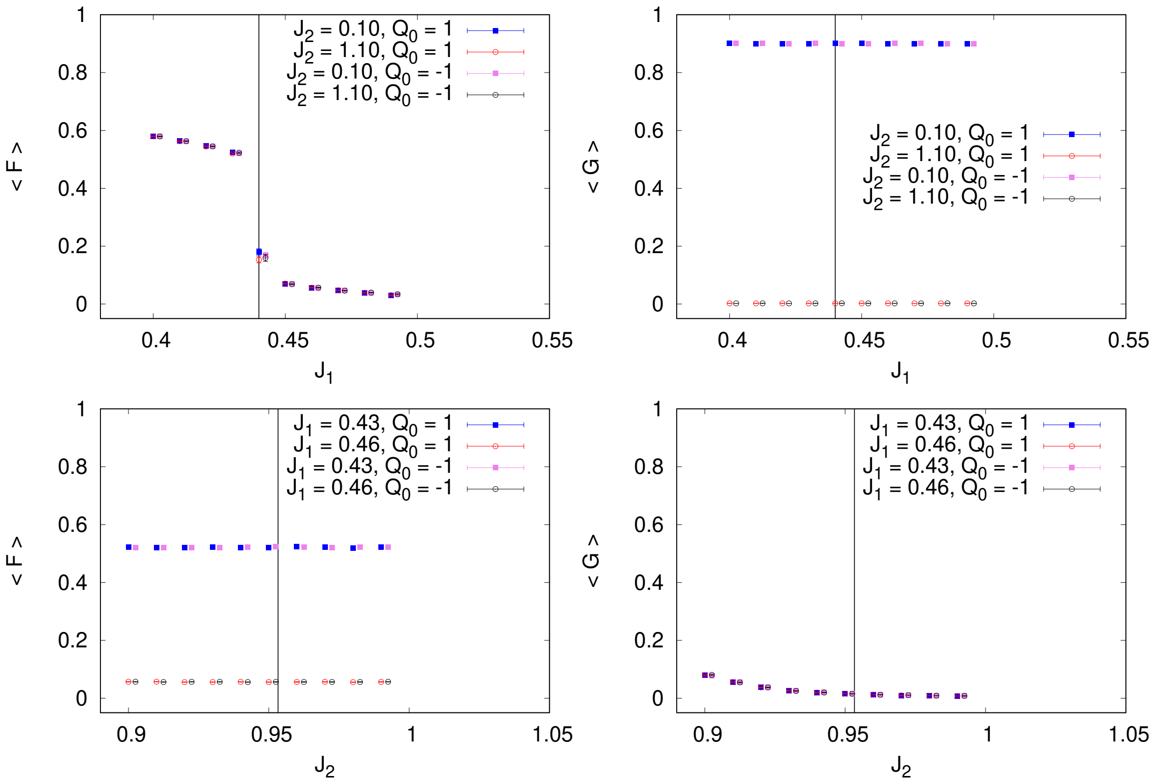

Figure 2. Dependence of the plaquette and cube observables on J_1 (upper row) and J_2 (lower row) coupling constants. As predicted by the factorization theorem, $\langle G\rangle$ does not depend on J_1 (upper right panel), whereas $\langle F\rangle$ does not depend on J_2 (lower left panel). $\langle F\rangle$ show a significant jump around the expected first order phase transition marked by the solid vertical black line on the upper, right panel. As far as $\langle G\rangle$ is concerned, lower right panel shows a rather smooth dependence and no significant signs of the expected second order phase transition marked again by the solid vertical line. Figure 3 demonstrates that indeed a second order phase transition happens around the expected $J_2^{\rm crit}$. In all the cases, results from both topological sectors are shown: $Q_0 = 1$ and $Q_0 = -1$, with $Q_\mu = 1$ for $\mu \neq 0$. The data points for the latter are shifted by 0.0025 along the x-axis in order to increase the plot readability.

our attention to non-local observables: Polyakov line and Polyakov planes. The latter, as opposed to the former, are not subject to the factorization theorem and hence are expected to have a non-trivial dependence on both J_1 and J_2.

3.3 Numerical results for non-local observables

In this section we discuss results for extended observables. We study in details two such observables: the (volume averaged) Polyakov line, P_0, winding around the x-direction (2.10) and the Polyakov plane P_{01}, winding around the x and y directions (2.16).

Our expectations for these observables in the four possible phases are based on considerations of the system of infinite size in directions perpendicular to the winding directions. We mimic that limit by taking $L_3 \to \infty$, which is the direction perpendicular to both P_0 and P_{01}. We discuss our numerical findings below.

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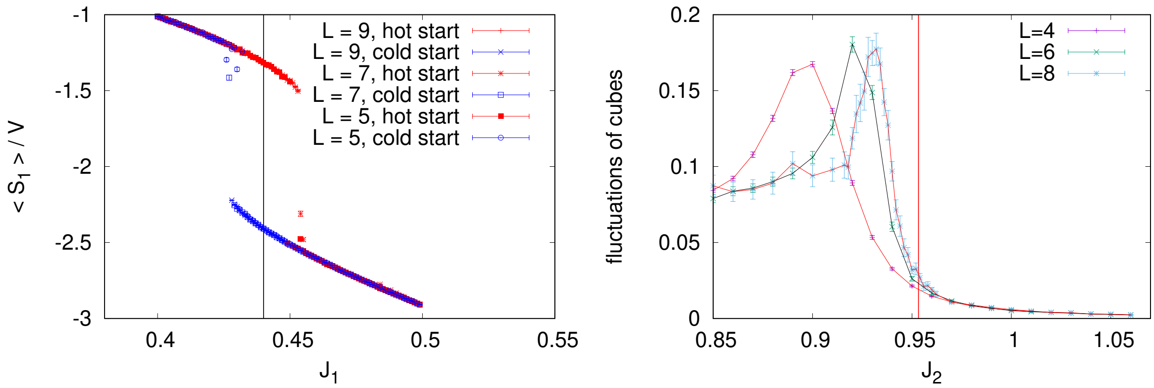

Figure 3. Left: results for the action around J_1 phase transition. There is a region of J_1 couplings where the simulations starting from different initial configurations: cold or hot converge to different local, meta-stable states. Outside of that region, the action has only one minimum and both simulations give the same average value of the plaquette action. Right: evidence for a second order phase transition in the J_2 coupling. Figure shows the fluctuations of the G observable for simulations at different linear size extends ranging from $L = 4$ up to $L = 8$. Data points shown are averages of independent simulations conducted in the $Q_0 = 1$ and $Q_0 = -1$ topological sectors. The maximum in the fluctuations approaches the theoretical, infinite limit value shown as the vertical line at J_2^{crit} as discussed around (2.31).

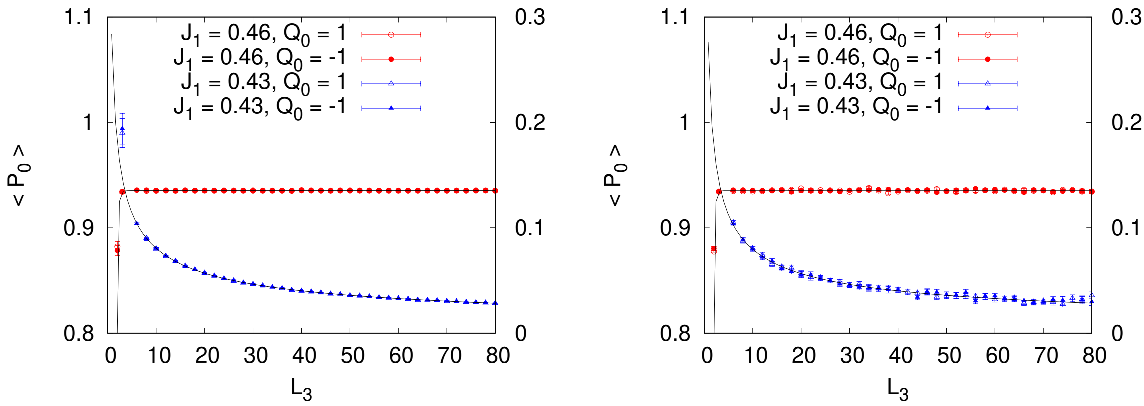

Figure 4. Demonstration of the dependence of the $\langle P_0 \rangle$ line on the transverse direction T for small $J_2 = 0.10$ and different $J_1 = 0.43$ and $J_1 = 0.46$ for both topological charges. Demonstration of the dependence of the $\langle P_0 \rangle$ line on the transverse direction L_3 for large $J_2 = 1.10$ and different $J_1 = 0.43$ and $J_1 = 0.46$ for both topological charges. The left axis shows the values of the data which has a constant nature, whereas the data sets falling towards zero have their values shown on the right axis.

We show the numerical results for $\langle P_0 \rangle$ and $\langle P_{01} \rangle$ in figures 4 and 5 at four pairs of coupling constants as a function of the extent of the lattice in the L_3 direction. In the left panels we gather results obtained at $J_1 = 0.43$ and $J_1 = 0.46$ at small $J_2 = 0.10$, whereas in the right panels we keep the same two values of J_1 but we change J_2 to a large value, $J_2 = 1.10$. As opposed to the previous section, where $\langle F \rangle$ and $\langle G \rangle$ were discussed as functions of J_1 and J_2 varying around their critical values, here we study the dependence on the L_3 extent at the four values of coupling constants selected in (3.5) and (3.6).

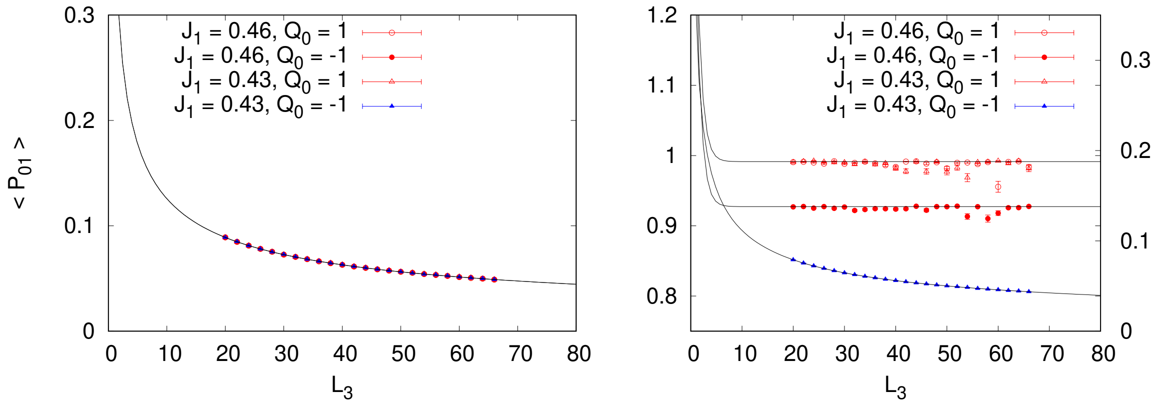

Figure 5. Demonstration of the $L_3^{-\frac{1}{2}}$ dependence of the $\langle P_{01}\rangle$ plane on the length L_3 of one transverse direction for small $J_2 = 0.10$ and different $J_1 = 0.43$ and $J_1 = 0.46$ for both topological charges. Demonstration of the dependence of the $\langle P_{01}\rangle$ plane on the transverse direction L_3 for large $J_2 = 1.10$ and different $J_1 = 0.43$ and $J_1 = 0.46$ for both values of the topological charge. The left axis shows the values of the data which has a constant nature, whereas the data sets falling towards zero have their values shown on the right axis.

We start with the discussion of Polyakov lines. The observable $\langle P_0\rangle$ is expected to satisfy the factorization theorem. Indeed, we find that its average value does not depend on the value of the J_2 coupling constant. As a consequence, the left and right panels of figure 4, showing the results for $J_2 = 0.10$ and $J_2 = 1.10$ respectively, look very similar. Two scenarios can be realized as the volume of the lattice grows: either the value of $\langle P_0\rangle$ decreases and ultimately vanishes in the infinite volume limit, or it becomes approximately constant for large volumes. Both scenarios are shown in figure 4: for $J_1 < J_1^{\mathrm{crit}}$ $\langle P_0\rangle$ decreases as $L_3^{-\frac{1}{2}}$ in the trivial and non-trivial topological sectors. On the contrary, for $J_1 > J_1^{\mathrm{crit}}$ we observe that $\langle P_0\rangle$ stays constant. Data points at very small volumes, $L_3 = 2$ and $L_3 = 3$, exhibit finite volume corrections which vanish rapidly with increasing volume. For $L_3 > 4$ a constant fit to the data with $J_1 > J_1^{\mathrm{crit}}$ and a fit with an Ansatz of the form $b + cL_3^{-\frac{1}{2}}$ with b,c being fit parameters to the data with $J_1 < J_1^{\mathrm{crit}}$, describe the data very well within their statistical uncertainties. This allows us to conclude that indeed the Polyakov line is a good order parameters for the phase transition in J_1 as it behaves differently on the different sides of J_1^{crit},

$$\langle P_0\rangle = 0 \text{ for } J_1 < J_1^{\mathrm{crit}},\ \text{any } J_2,\ \text{any } Q_0,\ L_3 \to \infty, \tag{3.7}$$

$$\langle P_0\rangle > 0 \text{ for } J_1 > J_1^{\mathrm{crit}},\ \text{any } J_2,\ \text{any } Q_0,\ L_3 \to \infty. \tag{3.8}$$

The situation with the Polyakov plane P_{01} is more complicated, as it depends non-trivially on both J_1 and J_2. Moreover this dependence is different in different topological sectors. We show the data in figure 5. Again, the left panel contains results for $J_2 < J_2^{\mathrm{crit}}$ while the right panel for $J_2 > J_2^{\mathrm{crit}}$. As opposed to the situation with Polyakov lines, now the plots are no longer similar and there is a nontrivial dependence on J_2. On the left panel, i.e. for small J_2, all data sets show a $L_3^{-\frac{1}{2}}$ dependence signaling that $\langle P_{01}\rangle$ vanishes in this region of phase space in the infinite volume limit. This happens no matter what

J_1	J_2	Q_0	$\langle P_{01}\rangle$
0.43	0.1	1	0.0631(2)
0.43	0.1	-1	0.0631(2)
0.46	0.1	1	0.0630(1)
0.46	0.1	-1	0.0630(1)
0.43	1.1	1	0.9815(3)
0.43	1.1	-1	0.0720(2)
0.46	1.1	1	0.9838(2)
0.46	1.1	-1	0.9238(1)

Table 1. Assembled average values of $\langle P_{01}\rangle$ in the four regions of phase diagram estimated on a lattice with $L = 4$ and $L_3 = 40$.

value of J_1 we chose and in both, trivial and non-trivial topological sectors. The right panel contains data for $J_2 > J_2^{\rm crit}$. Only a single data set, the blue one corresponding to $J_1 < J_1^{\rm crit}$ in the topologically charged sector $Q_0 = -1$, vanishes. In all remaining cases the data show a rather constant value as L_3 is increased, suggesting a non-zero value in the infinite volume limit. Looking from another perspective, in the trivial topological sector $Q_0 = 1$, $\langle P_{01}\rangle$ depends only on J_2, it vanishes for $J_2 < J_2^{\rm crit}$ and is nonzero for $J_2 > J_2^{\rm crit}$, irrespective of J_1. In the non-trivial topological sector, $\langle P_{01}\rangle$ vanishes in three corners of the phase space, except of the region where both J_1 and J_2 are large, i.e. $J_1 > J_1^{\rm crit}$ and $J_2 > J_2^{\rm crit}$. Hence, $\langle P_{01}\rangle$ at $Q_0 = -1$ is sensitive to both J_1 and J_2 and provides an order parameter for both phase transitions.

Summarizing, for $\langle P_{01}\rangle$ we have in the limit $L_3 \to \infty$:

$$\langle P_{01}\rangle = 0 \text{ for } \text{ any } J_1,\ J_2 < J_2^{\rm crit},\ \text{any } Q_0, \tag{3.9}$$
$$\langle P_{01}\rangle = 0 \text{ for } J_1 < J_1^{\rm crit},\ J_2 > J_2^{\rm crit},\ Q_0 = -1, \tag{3.10}$$
$$\langle P_{01}\rangle > 0 \text{ for } J_1 < J_1^{\rm crit}, J_2 > J_2^{\rm crit}, Q_0 = 1, \tag{3.11}$$
$$\langle P_{01}\rangle > 0 \text{ for } J_1 > J_1^{\rm crit}, J_2 > J_2^{\rm crit},\ \text{any } Q_0. \tag{3.12}$$

Distinction between phases is seen also by comparing values of $\langle P_{01}\rangle$ for different coupling constants at one finite value of L_3, see table 1.

4 Summary and conclusions

We have presented an explicit construction of a dynamical lattice model with a local symmetry based on a 2-group. It depends on two coupling constants J_1, J_2. We have analyzed the parameter space, first by using dualities to known simpler models, second by simulating the model numerically through Monte Carlo method. Theoretical discussion allows to designate four possible phases in the four corners of the coupling constant plane. In order to study the phase diagram quantitatively, we proposed several candidates for order parameters. Two proposals based on local observables, the average plaquette F and the average cube G, are sensitive to the phase transition only in one of the coupling constants. It

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follows from the factorization theorem, which we formulate and prove, that F constructed from link variables shows the phase transition in J_1, whereas G built out of faces shows the phase transition in J_2. The other two candidates for order parameters are non-local observables. Polyakov lines, which are products of link variables, again, feel only the phase transition in the J_1 coupling constant. Finally, the Polyakov plane exhibits a non-trivial dependence on both J_1 and J_2 and hence can be used as an order parameter for both phase transitions. Furthermore, its expectation value depends on the topological charge sector.

We would like to close this work by mentioning three problems for future study. Firstly, different techniques are required to perform averaging with respect to topological charge sectors. This is because Monte Carlo simulations performed in a fixed topological charge sector do not provide values of weights (partition functions) of distinct sectors. This difficulty is relevant only for those observables for which the average obtained in different topological charge sectors do not agree. The only observable with this property studied in this work is the Polyakov plane. Secondly, it would be interesting to obtain some results about extended surface observables on lattices of topology different than torus, perhaps also for more general crossed modules. Another intriguing question is whether there exists some natural construction of a dynamical higher gauge theory in which factorization theorem does not hold.

Acknowledgments

B. Ruba acknowledges the support of the SciMat grant U1U/P05/NO/03.39. Computer time allocations 'plgtmdlangevin2' and 'plgnnformontecarlo' on the Prometheus supercomputer hosted by AGH Cyfronet in Kraków, Poland was used through the polish PLGRID consortium.

A Non-spherical Wilson surfaces

In this appendix we use terminology and notations from [25]. Thus in contrast to the remainder of the paper, this part is not fully self-contained.

We consider field configurations on a connected CW-complex X valued in a crossed module $\mathbb{G} = (\mathcal{E}, \Phi, \Delta, \triangleright)$. They are described by homomorphisms $\Pi_2(X_2, X_1; X_0) \to \mathbb{G}$, resp. $\Pi_2(X, X_1; X_0) \to \mathbb{G}$ under the flatness constraint which is the minimization condition for the action S_2 from this paper. Replacing X_2 by X in the former case and choosing a base point $* \in X_0$, we are led to considering homomorphisms $\Pi_2(X, X_1; *) \to \mathbb{G}$. Given such a homomorphism, we obtain a commutative diagram of group homomorphisms

$$\begin{tikzcd} \Phi \arrow[d, "\Delta"'] & \pi_2(X, X_1, *) \arrow[l, "\varphi"'] \arrow[r, "h_2"] \arrow[d, "\partial"'] & H_2(X, X_1) \arrow[d, "\partial"] \\ \mathcal{E} & \pi_1(X_1, *) \arrow[l, "\epsilon"'] \arrow[r, "h_1"] & H_1(X_1) \end{tikzcd}$$

in which h_i are the Hurewicz homomorphisms. Hurewicz theorem and its relative version imply that h_i are surjective with $\ker(h_1)$ and $\ker(h_2)$ generated by expression of the form $\{\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}\}_{\gamma_1,\gamma_2\in\pi_1(X_1,*)}$ and $\{(\gamma \triangleright \sigma)\sigma^{-1}\}_{\substack{\gamma\in\pi_1(X_1,*)\\ \sigma\in\pi_2(X,X_1,*)}}$, respectively.

Now let $\sigma \in \pi_2(X, X_1, *)$ be an element such that $\partial(h_2(\sigma)) = 0$, i.e. such that the relative chain associated to σ is a cycle. Since $\partial \circ h_2 = h_1 \circ \partial$, we then have that $\partial\sigma$ belongs to $\ker(h_1)$. It follows that φ_σ belongs to the intersection of $\mathrm{im}(\Delta)$ and the commutant $[\mathcal{E}, \mathcal{E}]$ of $\mathcal{E}$. If this intersection is trivial (e.g. if $\mathcal{E}$ is abelian, which is satisfied by the crossed module featuring in the model considered in this paper), then $\epsilon_{\partial\sigma} = 1$, so $\varphi_\sigma \in \ker(\Delta)$. Under a gauge transformation

$$\varphi_\sigma \mapsto \xi_{b(\sigma)} \triangleright \left(\psi^{(\epsilon)}_{\partial\sigma}\,\varphi_\sigma\right). \tag{A.1}$$

If $\mathcal{E}$ acts trivially on $\ker(\Delta)$, factor $\xi_{b(\sigma)}$ may be omitted. We claim that furthermore $\psi^{(\epsilon)}_{\partial\sigma} = 1$. Indeed, since all ψ_e are in $\ker(\Delta)$ and $\mathcal{E}$ acts trivially on $\ker(\Delta)$, all epsilons present in the definition of $\psi^{(\epsilon)}_{\partial\sigma}$ may be omitted. On the other hand, since $\partial\sigma$ is a product of commutators, also $\psi^{(1)}_{\partial\sigma}$ is a product of commutators of elements in $\ker(\Delta)$, hence trivial ($\ker(\Delta)$ being abelian). Therefore under the running assumptions φ_σ is gauge-invariant, so it may be used as an observable.

It is interesting to ask whether φ_σ depends on the choice of σ representing the cycle $h_2(\sigma)$. If σ' is another representative of the same cycle, then $\sigma' = \sigma\sigma_0$ for some $\sigma_0 \in \ker(h_2)$. Thus $\varphi_{\sigma'} = \varphi_\sigma\varphi_{\sigma_0}$. We have to describe φ_{σ_0}. By the characterization of $\ker(h_2)$ given earlier we have that σ_0 is the product $\prod\limits_{i=1}^{n}(\gamma_i \triangleright \tau_i)\tau_i^{-1}$ for some $\gamma_i \in \pi_1(X_1, *)$ and $\tau_i \in \pi_2(X, X_1, *)$. Thus

$$\varphi_{\sigma_0} = \prod_{i=1}^{n}(\epsilon_{\gamma_i} \triangleright \varphi_{\tau_i})\varphi_{\tau_i}^{-1}. \tag{A.2}$$

This element is trivial if either of the following two conditions is satisfied:

- φ_{τ_i} are in $\ker(\Delta)$, i.e. $\epsilon_{\partial\tau_i}$ are trivial,
- ϵ_{γ_i} are elements of $\mathcal{E}$ which act trivially on Φ; if $\mathrm{im}(\Delta)$ acts trivially (which is satisfied in the model discussed in this paper), this is automatically satisfied if $\overline{\epsilon}$ is trivial.

In the language used in the main text, these two conditions correspond to $J_1 = \infty$ and trivial topological charge, respectively. Assuming that one of these conditions holds, we find that φ_σ depends on σ only through the corresponding homology class in $H_2(X)$ (respectively $H_2(X_2)$ if we do not assume flatness of φ).

B Comparison with continuous theories

We will now compare the model investigated in this paper with its counterparts in continuous field theory. These analogies, Wilson's construction of lattice gauge theories and simplicity are among our main motivations to focus on the action functional that we have chosen.

Let us start with algebraic preliminaries. A crossed module of Lie groups is a crossed module of groups $\mathbb{G} = (\mathcal{E}, \Phi, \Delta, \triangleright)$ such that $\mathcal{E}$ and Φ are Lie groups and $\Delta, \triangleright$ are smooth maps. By differentiation it gives rise to a crossed module of Lie algebras, which consists of

- Lie algebras $\mathfrak{e}$ and $\mathfrak{f}$ (Lie algebras of $\mathcal{E}$ and Φ),
- a Lie algebra homomorphism $\Delta : \mathfrak{f} \to \mathfrak{e}$,

- an action $\triangleright$ of $\mathfrak{e}$ on $\mathfrak{f}$ by derivations, i.e. a bilinear map $\mathfrak{e} \times \mathfrak{f} \to \mathfrak{f}$ satisfying

$$[e_1, e_2] \triangleright f_1 = e_1 \triangleright (e_2 \triangleright f_1) - e_2 \triangleright (e_1 \triangleright f_1), \quad e_1 \triangleright [f_1, f_2] = [e_1 \triangleright f_1, f_2] + [f_1, e_1 \triangleright f_2] \tag{B.1}$$

  for $e_1, e_2 \in \mathfrak{e}$ and $f_1, f_2 \in \mathfrak{f}$,

subject to two *Peiffer's identities*:

- $\Delta(e \triangleright f) = [e, \Delta f]$ for $e \in \mathfrak{e}$ and $f \in \mathfrak{f}$,
- $(\Delta f_1) \triangleright f_2 = [f_1, f_2]$ for $f_1, f_2 \in \mathfrak{f}$.

We will also need the version of $\triangleright$ differentiated in the second argument only. It is an action of the group $\mathcal{E}$ on the Lie algebra $\mathfrak{f}$ by homomorphisms, i.e.

$$(\epsilon_1 \epsilon_2) \triangleright f = \epsilon_1 \triangleright (\epsilon_2 \triangleright f), \qquad \epsilon_1 \triangleright [f_1, f_2] = [\epsilon_1 \triangleright f_1, \epsilon_1 \triangleright f_2] \tag{B.2}$$

for $\epsilon_1, \epsilon_2 \in \mathcal{E}$ and $f_1, f_2 \in \mathfrak{f}$. It satisfies its own version of one of Peiffer's identites:

$$\Delta(\epsilon \triangleright f) = \epsilon (\Delta f) \epsilon^{-1}, \tag{B.3}$$

in which the conjugation by ϵ should be read as the adjoint action of $\mathcal{E}$ on its Lie algebra.

For simplicity of presentation we restrict attention to gauge fields given by globally defined differential forms. This is sufficient in flat space, but on general manifolds one should consider fields defined on local coordinate patches, related by gauge transformations on the overlaps.

A crossed module-valued gauge field consists of a $\mathfrak{e}$-valued one-form field $A = A_\mu \mathrm{d}x^\mu$ and a $\mathfrak{f}$-valued two-form field $B = \frac{1}{2} B_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$, subject to the fake flatness constraint:

$$\Delta B = \mathrm{d}A + \frac{1}{2}[A, A]. \tag{B.4}$$

The right hand side of this equation is the standard field strength tensor (or curvature 2-form) F built from the A field. It satisfies the Bianchi identity:

$$\mathrm{d}_A F = 0, \tag{B.5}$$

where $\mathrm{d}_A = d + A$ is the (exterior) covariant derivative. We will consider the 3-form field

$$G := \mathrm{d}_A B = \mathrm{d}B + A \triangleright B, \tag{B.6}$$

also referred to as (higher) field strength tensor. As a consequence of fake flatness and Peiffer's identities, it satisfies

$$\mathrm{d}_A G = F \triangleright B = \Delta B \triangleright B = [B, B] = 0. \tag{B.7}$$

Gauge fields are subject to two types of gauge transformations, both preserving the fake flatness constraint. Firstly, for a function ξ valued in the group $\mathcal{E}$ we have the transformation

$$A \mapsto \xi A \xi^{-1} + \xi \mathrm{d}\xi^{-1}, \tag{B.8a}$$

$$B \mapsto \xi \triangleright B. \tag{B.8b}$$

Under these transformations we have

$$F \mapsto \xi F \xi^{-1}, \tag{B.9a}$$
$$G \mapsto \xi \rhd G. \tag{B.9b}$$

Second type of transformations is parametrized by 1-forms ψ valued in $\mathfrak{f}$. It is given by

$$A \mapsto A + \Delta\psi, \tag{B.10a}$$
$$B \mapsto B + \mathrm{d}_A\psi + \frac{1}{2}[\psi,\psi]. \tag{B.10b}$$

Transformation laws for fields strength tensors take the form

$$F \mapsto F + \Delta\left(\mathrm{d}_A\psi + \frac{1}{2}[\psi,\psi]\right), \tag{B.11a}$$
$$G \mapsto G. \tag{B.11b}$$

Equation (B.11a) means that for general ψ the field F changes in a complicated (non-linear) way. However, it is invariant if ψ is assumed to be valued in the (normal) Lie subalgebra $\ker(\Delta) \subset \mathfrak{f}$. Therefore we choose to regard only those ψ transformations as gauge redundancies. That is, fields related by transformations (B.10) with Δ not in $\ker(\Delta)$ are deemed physically inequivalent. Then one may obtain a generalization of Yang-Mills theory, with two standard local observables: F and G.

The self-evident generalization of the (Euclidean) Yang-Mills Lagrangian depends on two coupling constants g, g' and takes the form

$$\mathcal{L} = \frac{1}{4g^2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle_{\mathfrak{e}} + \frac{1}{12g'^2}\langle G_{\alpha\beta\gamma}, G^{\alpha\beta\gamma}\rangle_{\mathfrak{f}}. \tag{B.12}$$

Here $\langle\cdot,\cdot\rangle_{\mathfrak{e}}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{f}}$ are bilinear forms on $\mathfrak{e}$ and $\mathfrak{f}$ invariant under the action of $\mathcal{E}$, as required by the demand of gauge invariance. If unrestricted gauge transformations (B.10) were admitted, the first term would have to be skipped, corresponding to the limit $g \to \infty$ ($J_1 \to 0$ in the notation of the main part of the text).

Two terms of the action (2.3) are the most natural analogues of the two terms of (B.12) in the setting of lattice spacetime and the particular crossed module of discrete groups, much the same way as the standard lattice $\mathbb{Z}_2$ gauge theory action may be seen as an analogue of the Yang-Mills action.

Besides the two terms of (B.12), one could contemplate including other:

- Contraction of F and G tensors is impossible because the number of indices does not match. There might exist terms involving more than two field strength tensors, such as $\langle F_{\mu\nu}, F^{\mu\nu}\rangle_{\mathfrak{e}}\langle G_{\alpha\beta\gamma}, G^{\alpha\beta\gamma}\rangle_{\mathfrak{f}}$. Such term is of high (naive) dimension and involves four derivatives, making it rather suspect from field theoretic point of view.

- There could exist interesting θ or Chern-Simons type terms, which are beyond the scope of this work.

References

[1] M. Henneaux and C. Teitelboim, *p-Form electrodynamics*, *Found. Phys.* **16** (1986) 593 [INSPIRE].

[2] J.C. Baez, *Higher Yang-Mills theory*, hep-th/0206130 [INSPIRE].

[3] H. Pfeiffer, *Higher gauge theory and a non-Abelian generalization of 2-form electrodynamics*, *Annals Phys.* **308** (2003) 447 [hep-th/0304074] [INSPIRE].

[4] J.C. Baez and J. Huerta, *An invitation to higher gauge theory*, *Gen. Rel. Grav.* **43** (2011) 2335 [arXiv:1003.4485] [INSPIRE].

[5] J.C. Baez and A.D. Lauda, *Higher dimensional algebra. V:2-groups*, *Theor. Appl. Categ.* **12** (2004) 423 [math/0307200].

[6] R. Brown, P.J. Higgins and R. Sivera, *Nonabelian Algebraic Topology. Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids*, with contributions by Ch. D. Wensley and S.V. Soloviev, volume 15, Zürich, European Mathematical Society (EMS) (2011).

[7] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized global symmetries*, *JHEP* **02** (2015) 172 [arXiv:1412.5148] [INSPIRE].

[8] S. Gukov and A. Kapustin, *Topological Quantum Field Theory, Nonlocal Operators, and Gapped Phases of Gauge Theories*, arXiv:1307.4793 [INSPIRE].

[9] A. Kapustin and R. Thorngren, *Topological field theory on a lattice, discrete theta-angles and confinement*, *Adv. Theor. Math. Phys.* **18** (2014) 1233 [arXiv:1308.2926] [INSPIRE].

[10] A. Kapustin and R. Thorngren, *Higher Symmetry and Gapped Phases of Gauge Theories*, *Prog. Math.* **324** (2017) 177.

[11] D.N. Yetter, *TQFT's from homotopy 2-types*, *J. Knot Theor. Ramif.* **02** (1993) 113.

[12] T. Porter, *Topological Quantum Field Theories from Homotopy n-Types*, *J. Lond. Math. Soc.* **58** (1998) 723.

[13] J.F. Martins and T. Porter, *On Yetter's Invariant and an Extension of the Dijkgraaf-Witten Invariant to Categorical Groups*, *Theor. Appl. Categ.* **18** (2007) 118 [math/0608484].

[14] F. Girelli, H. Pfeiffer and E.M. Popescu, *Topological higher gauge theory: From BF to BFCG theory*, *J. Math. Phys.* **49** (2008) 032503 [arXiv:0708.3051] [INSPIRE].

[15] D.J. Williamson and Z. Wang, *Hamiltonian models for topological phases of matter in three spatial dimensions*, *Annals Phys.* **377** (2017) 311 [arXiv:1606.07144] [INSPIRE].

[16] M. Atiyah, *Topological quantum field theories*, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1989) 175 [INSPIRE].

[17] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [INSPIRE].

[18] Y.-A. Chen and A. Kapustin, *Bosonization in three spatial dimensions and a 2-form gauge theory*, *Phys. Rev. B* **100** (2019) 245127 [arXiv:1807.07081] [INSPIRE].

[19] Y.-A. Chen, *Exact bosonization in arbitrary dimensions*, *Phys. Rev. Res.* **2** (2020) 033527.

JHEP09(2021)068

[20] A. Bochniak and B. Ruba, *Bosonization based on Clifford algebras and its gauge theoretic interpretation*, JHEP **12** (2020) 118 [arXiv:2003.06905] [INSPIRE].

[21] S. Palmer and C. Sämann, *The ABJM model is a higher gauge theory*, Int. J. Geom. Meth. Mod. Phys. **11** (2014) 1450075.

[22] A. Bullivant, M. Calçada, Z. Kádár, P. Martin and J.F. Martins, *Topological phases from higher gauge symmetry in $3+1$ dimensions*, Phys. Rev. B **95** (2017) 155118 [arXiv:1606.06639] [INSPIRE].

[23] C. Delcamp and A. Tiwari, *From gauge to higher gauge models of topological phases*, JHEP **10** (2018) 049 [arXiv:1802.10104] [INSPIRE].

[24] A. Bullivant, M. Calçada, Z. Kádár, J.F. Martins and P. Martin, *Higher lattices, discrete two-dimensional holonomy and topological phases in $(3+1)D$ with higher gauge symmetry*, Rev. Math. Phys. **32** (2020) 2050011.

[25] A. Bochniak, L. Hadasz and B. Ruba, *Dynamical generalization of Yetter's model based on a crossed module of discrete groups*, JHEP **03** (2021) 282 [arXiv:2010.00888] [INSPIRE].

[26] F.J. Wegner, *Duality in Generalized Ising Models and Phase Transitions Without Local Order Parameters*, J. Math. Phys. **12** (1971) 2259 [INSPIRE].

[27] H.A. Kramers and G.H. Wannier, *Statistics of the Two-Dimensional Ferromagnet. Part I*, Phys. Rev. **60** (1941) 252 [INSPIRE].

[28] F.J. Wegner, *Flow-equations for Hamiltonians*, Annalen Phys. **3** (1994) 77.

[29] S. Akiyama, Y. Kuramashi, T. Yamashita and Y. Yoshimura, *Phase transition of four-dimensional Ising model with higher-order tensor renormalization group*, Phys. Rev. D **100** (2019) 054510 [arXiv:1906.06060] [INSPIRE].

[30] P.H. Lundow and K. Markström, *Critical behavior of the Ising model on the four-dimensional cubic lattice*, Phys. Rev. E **80** (2009) 031104.

[31] N. Metropolis and S. Ulam, *The Monte Carlo Method*, J. Am. Statist. Assoc. **44** (1949) 335.

[32] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller and E. Teller, *Equation of State Calculations by Fast Computing Machines*, J. Chem. Phys. **21** (1953) 1087 [INSPIRE].

[33] W.K. Hastings, *Monte Carlo sampling methods using Markov chains and their applications*, Biometrika **57** (1970) 97.

[34] M. Creutz, *Monte Carlo study of quantized* SU(2) *gauge theory*, Phys. Rev. D **21** (1980) 2308 [INSPIRE].

[35] S.L. Adler, *Over-relaxation method for the Monte Carlo evaluation of the partition function for multiquadratic actions*, Phys. Rev. D **23** (1981) 2901 [INSPIRE].

[36] C. Whitmer, *Over-relaxation methods for Monte Carlo simulations of quadratic and multiquadratic actions*, Phys. Rev. D **29** (1984) 306 [INSPIRE].

[37] F.R. Brown and T.J. Woch, *Overrelaxed heat-bath and Metropolis algorithms for accelerating pure gauge Monte Carlo calculations*, Phys. Rev. Lett. **58** (1987) 2394 [INSPIRE].

[38] S.L. Adler, *Overrelaxation algorithms for lattice field theories*, Phys. Rev. D **37** (1988) 458 [INSPIRE].

[39] M. Creutz, L. Jacobs and C. Rebbi, *Experiments with a Gauge-Invariant Ising System*, Phys. Rev. Lett. **42** (1979) 1390 [INSPIRE].

Published for SISSA by Springer

Received: *April 13, 2020*
Revised: *November 3, 2020*
Accepted: *November 6, 2020*
Published: *December 18, 2020*

Bosonization based on Clifford algebras and its gauge theoretic interpretation

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Abstract: We study the properties of a bosonization procedure based on Clifford algebra valued degrees of freedom, valid for spaces of any dimension. We present its interpretation in terms of fermions in presence of $\mathbb{Z}_2$ gauge fields satisfying a modified Gauss' law, resembling Chern-Simons-like theories. Our bosonization prescription involves constraints, which are interpreted as a flatness condition for the gauge field. Solution of the constraints is presented for toroidal geometries of dimension two. Duality between our model and $(d-1)$-form $\mathbb{Z}_2$ gauge theory is derived, which elucidates the relation between the approach taken here with another bosonization map proposed recently.

Keywords: Gauge Symmetry, Lattice Quantum Field Theory, Topological States of Matter, Chern-Simons Theories

ArXiv ePrint: 2003.06905

https://doi.org/10.1007/JHEP12(2020)118

JHEP12(2020)118

Contents

1 Introduction

Many fermionic systems admit bosonizations, i.e. alternative descriptions formulated using bosonic operators. Such correspondences are especially abundant for theories formulated in spacetime dimension two [1–3]. Their importance stems from the fact that they allow to construct analytic solutions of certain models [4, 5], to gain nonperturbative insights into dynamics of strongly coupled systems [6] and, more recently, to understand certain phases of topologically nontrivial fermionic matter [7]. Furthermore, there exist systems for which dualities help to overcome problems in numerical studies, such as the sign problem in Monte Carlo simulations [8–10] or difficulties in implementation of operators acting on Hilbert spaces which do not factorize into tensor products of on-site Hilbert spaces. This last problem may also have some significance for the field of quantum information [11, 12].

The most well-known bosonization methods apply only to $1+1$-dimensional systems. Some proposals valid in higher dimensions have been put forward [7, 13–23]. See also reviews in [24–27]. Each of these constructions involves some difficulties not present for two-dimensional systems, such as non-locality or presence of complicated constraints, often interpreted as the Gauss' law of some gauge theory. One could argue that this is an inherent feature of models involving fermionic degrees of freedom. Further study of these phenomena might help to eventually construct bosonization maps more suitable for practical calculations, which is the main motivation of this work.

The main part of this paper is concerned with the study of the bosonization method proposed in [15]. In this approach fermion fields are replaced by on-site Euclidean Γ matrices. For this reason we call it the Γ model. This model is bosonic in the sense that the Γ matrices, which serve as its elementary fields, commute when placed on distinct lattice sites. Moreover its Hilbert space is the tensor product of Hilbert spaces associated to individual lattice sites. The price to pay for this convenience is the necessity to introduce certain constraints on physical states. Correspondence between the Γ model with constraints and fermions, at least for the free fermion hamiltonian, has been conjectured based on a comparison between relations satisfied by operators present in hamiltonians of these two models. Precise statement of this correspondence has been formulated and proven for the first time in [28]. It turned out that the proposed bosonization map is valid for any hamiltonian, hence purely kinematical. Here we extend it by considering more general geometries. We provide a new proof of validity of this construction, inspired by techniques from [13]. Furthermore, we provide a new interpretation of constraints present in the Γ model as the pure gauge condition for a certain $\mathbb{Z}_2$ gauge field. We show that fermions coupled to general $\mathbb{Z}_2$ gauge fields can be modeled by modifying the form of constraints, without altering the form of the bosonized hamiltonian. The full Hilbert space of the Γ model decomposes into a direct sum of subspaces corresponding to all possible gauge fields. This decomposition has the interesting property that only states with specific fermionic parity, depending on the gauge field, are present. We illustrate the main features of our model by presenting examples in the cases of a specific geometry (two-dimensional tori, for which we also solve the constraints) and for a simple class of solvable fermionic hamiltonians. This work parallels [29], which motivated our studies, allowed to formulate initial hypotheses and test them using symbolic algebra software.

It is natural to ask whether it is possible to make the gauge field present in the Γ model dynamical. In other words, does the Γ model with no constraints imposed provide a bosonization of a some theory of fermions coupled to a $\mathbb{Z}_2$ gauge field? We show that such mapping does indeed exist. It is local for even fermionic operators and for gauge field operators of magnetic type,[1] but operators involving the electric field are represented in a complicated way, which depends on a choice of a loop wrapping around the whole lattice. Similarly, the elementary field of the Γ model is non-local on the gauge theory side.

Gauge theory corresponding to the unconstrained Γ model involves a mechanism present in the Dijkgraaf-Witten theory [30–32] and more general gauge theories with Chern-

[1]We call an operator magnetic if it is a function of the gauge field on a single time slice and electric if it acts by flipping the gauge field. General observables in gauge theory involve operators of both types.

Simons-like topological terms: Hilbert space representation of time-independent gauge transformations, here written for simplicity in the U(1) continuum theory language,[2]

$$|A_i\rangle \mapsto |A_i + \partial_i \theta\rangle \tag{1.1}$$

are modified by introducing gauge field dependent phase factors:

$$|A_i\rangle \mapsto e^{iI(\theta,A)}|A_i + \partial_i \theta\rangle. \tag{1.2}$$

This has the consequence that the Gauss' law is altered, which leads to a deformation of the algebra of gauge-invariant operators. In particular, the constraint on the total charge, obtained by integrating the Gauss' law over the whole space, is modified. This is the celebrated flux attachment mechanism [33]: electric excitations in models of this type are decorated by magnetic fields. Braiding of two such excitations involves Aharonov-Bohm phases, leading to a transmutation of statistics. In our case, the total number of fermions modulo two becomes related to the value of a certain magnetic observable. An unpleasant feature of the gauge theory corresponding to the Γ model is that the functional $I(\theta, A)$ in (1.2) depends non-locally on the gauge field A. We demonstrate that under certain assumptions about the lattice this non-locality may be removed by a canonical transformation which preserves the form of all fermionic and magnetic observables (so bosonization is still local for those operators for which it initially was).

There exists a duality mapping which relates the Γ model to higher gauge theories proposed in the context of bosonization in [13, 14, 34]. In some aspects it resembles the classical Kramers-Wannier duality [35]. It is clear that this correspondence has to involve a transition to the dual spatial lattice. Indeed, in our model local degrees of freedom act on Hilbert spaces associated to lattice sites, just as in the initial fermionic theory, while constraint operators are located on plaquettes. In the latter case, for spacetimes of dimension $d+1$, degrees of freedom associated to $(d-1)$-cells have been proposed, with fermionic operators placed on d-simplices and constraints on $(d-2)$-simplices. This setup has the advantage that it is naturally interpreted in terms of $(d-1)$-form gauge theory (involving the flux attachment mechanism). On the other hand, our formulation is more uniform, in the sense that it applies in unchanged form in any dimension. The amount of redundancy in the two approaches (defined as the ratio of the dimension of the full Hilbert space and the subspace defined by constraints) is the same order (and rather large) in both cases. Secondly, in our construction it is crucial that each lattice vertex is incident to an even number of edges. We remark here that it is possible to define the Γ model even if this condition is not satisfied, but in this case it is found to contain additional degrees of freedom, resembling Majorana fermions. This feature is discussed in the appendix B.

The organization of this paper is as follows. In section 2 we recall basic geometric concepts used in the main text. Reader not at all familiar with this language may want to consult introductory books in algebraic topology (see e.g. [36]) first. Section 3 is concerned mainly with the review of a known description of the algebra of even fermionic operators in terms of a convenient set of generators and relations. The main part of the

[2] Precise formulation suitable for our lattice models is given in the main text.

text starts in section 4. In subsections 4.1, 4.2 we define the Γ model and establish its correspondence with fermions. Then we derive the gauge-theoretic interpretation of this model in subsection 4.3. Presented constructions are illustrated by the example of toroidal geometry, discussed in the subsection 4.4 and the discussion of quadratic hamiltonians in 4.5. In the special case of dimension $2+1$ we solve the constraints relevant for our bosonization procedure and relate them to ground states of the Kitaev's toric code [11]. Section 5 is devoted to the study of modified gauge theories. Proof of the equivalence between the gauge theory proposed in the subsection 4.3 and the Γ model is presented in the subsection 5.1. Afterwards a generalization of this gauge model, involving modified Gauss' operators, is introduced in the subsection 5.2. We classify these theories up to equivalence given by (in general non-local) canonical transformations. This allows to find a local formulation of the gauge theory corresponding to the Γ model in the subsection 5.3. Afterwards, in section 6, we present the duality between the Γ model and higher gauge theory. This includes a brief discussion of the role of spin structures. We summarize in section 7. The paper is closed with two appendices. Appendix A is concerned with Heisenberg groups and their automorphisms for $\mathbb{Z}_2$-valued degrees of freedom, while appendix B discusses the extension of the Γ model to the case in which some vertices are incident to an odd number of edges.

2 Geometric setup

For any finite set S we let $|S|$ be the number of elements of S.

All physical systems will be considered on a connected graph $\mathfrak{G}=(V,E)$, which may (but does not have to) be the set of vertices and edges of a triangulation or more general cell decomposition of some manifold. We will assume that the graph $\mathfrak{G}$ is such that every edge connects two distinct vertices. Multiple edges which connect the same vertices are allowed. We let E_{or} be the set of oriented edges. Thus every edge $e \in E$ corresponds to two distinct elements of E_{or}. We have functions $s,t : E_{\mathrm{or}} \to V$, called source and target maps, which assign to $e \in E_{\mathrm{or}}$ its initial and final vertex, respectively. Furthermore, for every $e \in E_{\mathrm{or}}$ we let $\overline{e}$ be the same edge with its orientation reversed, so that $s(\overline{e})=t(e)$ and $t(\overline{e})=s(e)$. If $v=s(e)$ or $v=t(e)$, we say that e contains v and write $v \in e$. The star $\mathrm{St}(v)$ of a vertex $v \in V$ is defined as the set of all $e \in E$ which contain v. Number $\deg(v):=|\mathrm{St}(v)|$ is called the degree of v.

In order to keep track of various signs we shall use the language of chains, which are formal sums of geometric objects with coefficients in the field $\mathbb{Z}_2$ (integers modulo 2). More precisely, C_0 and C_1 are defined as the $\mathbb{Z}_2$-vector spaces with bases V and E, respectively. Linear map $\partial : C_1 \to C_0$, called the boundary operator, is defined first on basis elements by $\partial e = \sum\limits_{v \in e} v$. Its kernel (called the set of cycles) and image (called the set of boundaries) are denoted by Z_1 and B_0, respectively. There are perfect bilinear pairings $C_p \times C_p \to \mathbb{Z}_2$, given by $(v,v')=\delta_{v,v'}$ and $(e,e')=\delta_{e,e'}$. This allows to identify chain groups C_p with cochain groups $C^p := \mathrm{Hom}(C_p,\mathbb{Z}_2)$. Coboundary operator $C^0 \to C^1$ is defined as the adjoint of ∂, i.e. by $(\delta\epsilon,\tau)=(\epsilon,\partial\tau)$ for $\epsilon \in C^0$ and $\tau \in C_1$. Equivalently, $\delta v = \sum\limits_{v \in e} e$. Kernel and image of δ are denoted by Z^0 and B^1 and called the set of cocycles and the set of coboundaries, respectively. By construction, cocycles are orthogonal to boundaries, while coboundaries are orthogonal to cycles. In particular, there is an induced non-degenerate pairing $Z_1^* \times Z_1 \to \mathbb{Z}_2$, where $Z_1^* := C^1/B^1$. Thus Z_1^* may be identified with the dual space of Z_1. The image in Z_1^* of an element of $A \in C^1$ will be denoted by $[A]$.

For future reference we calculate the dimension of Z_1 (and hence also of Z_1^*) over $\mathbb{Z}_2$. In general the dimension of the domain of a linear operator is the sum of dimensions of the kernel and the range. Applying this to ∂ we obtain $\dim(Z_1) = \dim(C_1) - \dim(B_0)$. Connectedness of $\mathfrak{G}$ means that $\dim(B_0) = \dim(C_0) - 1$. Therefore

$$\dim(Z_1) = \dim(C_1) - \dim(C_0) + 1 = |E| - |V| + 1. \tag{2.1}$$

This means that each of sets Z_1 and Z_1^* has $2^{|E|-|V|+1}$ elements.

Tuple of oriented edges $\ell = (e_1, \dots, e_n)$ will be called a path if $t(e_i) = s(e_{i+1})$ for $i < n$. We will say that ℓ is a circuit if $t(e_n) = s(e_1)$. For every path ℓ we let $[\ell] = \sum_{i=1}^{n} e_i \in C_1$, where we forget the orientations of e_i. Chain $[\ell]$ is a cycle if and only if ℓ is a circuit. Circuit ℓ is said to be Eulerian if every edge $e \in E$ occurs exactly once among $e_1, \dots, e_n$. For every such circuit we have $[\ell] = \sum_{e \in E} e$. It is a classical result [37, section 4.2.1] in graph theory that Eulerian circuit exists if and only if every vertex has even degree. Clearly, the latter condition is equivalent to closedness of the chain $\zeta := \sum_{e \in E} e \in C_1$, i.e. to $\partial\zeta = 0$.

In some parts of this work (not essential for the main construction) we will have to assume that besides vertices and edges, the considered lattice is also equipped with a set of faces F, which are polygons whose sides are identified with edges. This allows to define the space of 2-chains C_2 with an obvious boundary map $\partial : C_2 \to C_1$. Its kernel and image are denoted by Z_2 and B_2, respectively. Homology group H_1 is defined as the quotient Z_1/B_1. There is also a scalar product $C_2 \times C_2 \to \mathbb{Z}_2$ given by $(f, f') = \delta_{f,f'}$ for $f, f' \in F$. Dualizing, there is also a coboundary map $\delta : C^1 \to C^2$ with kernel and image Z^1, B^2. Cohomology group $H^1 = Z^1/B^1$ is the dual space of H_1.

3 Fermions — generators and relations

Here we consider a specific class of fermionic models, defined below. We emphasize those properties that are used to prove validity of our bosonization prescription. In particular, we describe the algebra of even fermionic operators in terms of generators and relations. This result is similar to one in [13], with the statement and the proof adjusted to the fact that we work with finite, not necessarily simply-connected lattices. Our considerations are independent of dynamics, so we do not focus on any particular hamiltonian. In most of this section we repeat well-known facts, to some extent to fix notation.

First, let us denote by $\mathcal{A}$ the complex $*$-algebra generated by elements $\phi^*(v)$ and $\phi(v)$ (called creation and annihilation operators located at the vertex v) with $v \in V$, subject to the canonical anticommutation relations

$$\{\phi(v), \phi(v')\} = \{\phi^*(v), \phi^*(v')\} = 0, \qquad \{\phi(v), \phi^*(v')\} = \delta_{v,v'}. \tag{3.1}$$

By construction, every element of $\mathcal{A}$ may be written down as a linear combination of products of creation and annihilation operators. It is often useful to use a different set of generators of $\mathcal{A}$, e.g. the so-called Majorana operators:

$$X(v) = \phi(v) + \phi^*(v), \qquad Y(v) = i(\phi(v) - \phi^*(v)). \tag{3.2}$$

Defining relations (3.1) are equivalent to

$$\{X(v), Y(v')\} = 0, \qquad \{X(v), X(v')\} = \{Y(v), Y(v')\} = 2\delta_{v,v'}. \tag{3.3}$$

This shows that $\mathcal{A}$ is a Clifford algebra on $2|V|$ generators, and hence it is isomorphic to $\mathrm{End}(\mathcal{F})$, the algebra of linear operators on the unique (up to isomorphism) irreducible representation $\mathcal{F}$ of $\mathcal{A}$. Dimension of $\mathcal{F}$ is equal to $2^{|V|}$. Every finite-dimensional representation of $\mathcal{A}$ is a direct sum of finitely many copies of the irreducible representation.

Representation $\mathcal{F}$ is, of course, the Fock space. It is a Hilbert space with a distinguished element $|0\rangle$ (called the vacuum state), determined uniquely up to phase by the conditions $\phi(v)|0\rangle = 0$ and $\langle 0|0\rangle = 1$. Other states, labeled by $\mathbb{Z}_2$-valued 0-chains ϵ, are defined by acting with creation operators on the vacuum:

$$|\epsilon\rangle = \prod_{v \in V} \phi^*(v)^{(\epsilon,v)}|0\rangle. \tag{3.4}$$

This element depends on the ordering of vertices in the product, but different orderings give rise to states differing only by a factor ± 1. To well-define vectors $|\epsilon\rangle$, fix any total order on V once and for all. The set of all vectors $|\epsilon\rangle$ is an orthonormal basis of $\mathcal{F}$.

Let us define the grading element of $\mathcal{A}$:

$$\gamma = \prod_{v \in V}(1 - 2\phi^*(v)\phi(v)). \tag{3.5}$$

It satisfies $\gamma = \gamma^* = \gamma^{-1}$. For each $\alpha \in \mathbb{Z}_2$ we define

$$\mathcal{F}_\alpha = \{\psi \in \mathcal{F} |\ \gamma\psi = (-1)^\alpha \psi\}, \tag{3.6a}$$
$$\mathcal{A}_\alpha = \{T \in \mathcal{A} |\ \gamma T = (-1)^\alpha T\gamma\}. \tag{3.6b}$$

$\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$. Its action on $\mathcal{F}$ has two nontrivial invariant subspaces: $\mathcal{F}_0$ and $\mathcal{F}_1$, which are both of dimension $2^{|V|-1}$. It follows from the Artin-Weddeburn theory [38] that the algebra $\mathcal{A}_0$ is semisimple, with two simple factors $\mathcal{A}_{\alpha\alpha} = \mathrm{End}_{\mathbb{C}}(\mathcal{F}_\alpha)$, $\alpha \in \mathbb{Z}_2$. This means that every finite-dimensional representation V of $\mathcal{A}_0$ is isomorphic to $\bigoplus_{\alpha \in \mathbb{Z}_2} \mathcal{F}_\alpha^{\oplus [V:\mathcal{F}_\alpha]}$, where multiplicity $[V : \mathcal{F}_\alpha]$ is given by the formula

$$[V : \mathcal{F}_\alpha] = \frac{1}{\dim_{\mathbb{C}}(\mathcal{F}_\alpha)} \mathrm{tr}_V \left(\frac{1 + (-1)^\alpha \gamma}{2} \right). \tag{3.7}$$

The even subalgebra $\mathcal{A}_0$ is of our main interest here. It is easy to see that it is generated by elements $\{\gamma(v)\}_{v \in V}$ and $\{\mathfrak{s}(e)\}_{e \in E_{\mathrm{or}}}$, defined by

$$\gamma(v) = 1 - 2\phi^*(v)\phi(v), \qquad \mathfrak{s}(e) = X(s(e))X(t(e)). \tag{3.8}$$

We refer to $\gamma(v)$ and $\mathfrak{s}(e)$ as fermionic parity and kinetic operators, respectively. We will now give a complete set of relations satisfied by our chosen generators.[3] Firstly,

$$\gamma(v) = \gamma(v)^* = \gamma(v)^{-1}, \qquad \gamma(v)\gamma(v') = \gamma(v')\gamma(v), \tag{3.9a}$$

$$-\mathfrak{s}(e) = \mathfrak{s}(\bar{e}) = \mathfrak{s}(e)^* = \mathfrak{s}(e)^{-1}, \qquad \mathfrak{s}(e)\mathfrak{s}(e') = (-1)^{(\partial e, \partial e')}\mathfrak{s}(e')\mathfrak{s}(e), \tag{3.9b}$$

$$\gamma(v)\mathfrak{s}(e) = (-1)^{(\partial e, v)}\mathfrak{s}(e)\gamma(v). \tag{3.9c}$$

The final relation in $\mathcal{A}_0$ may be formulated as follows: if $\ell = (e_1, \dots, e_n)$ is a circuit, then

$$\mathfrak{s}(e_1) \cdot \ldots \cdot \mathfrak{s}(e_n) = 1. \tag{3.10}$$

Not all of these relations are independent. Indeed, suppose that some algebra $\mathcal{B}$ contains elements $\gamma(v)$ and $\mathfrak{s}(e)$ satisfying (3.9) and such that (3.10) holds for some circuits $\{\ell_i\}_{i=1}^{s}$ such that $[\ell_i]$ generate Z_1. Then for any circuit $\ell = (e_1, \dots, e_n)$ there exist coefficients c_i such that $[\ell] = \sum_{i=1}^{s} c_i [\ell_i]$. Using relations (3.9) and (3.10) for ℓ_i we obtain $\mathfrak{s}(e_1) \cdot \ldots \cdot \mathfrak{s}(e_n) = \pm 1$. The same calculation can be repeated in $\mathcal{A}_0$, so the sign on right hand side has to be $+1$, because (3.10) holds for all circuits in this case. Hence (3.10) is satisfied in $\mathcal{B}$ for all circuits ℓ.

In the rest of this section we will show that there are no other relations, i.e. that (3.9) and (3.10) generate all relations in $\mathcal{A}_0$. It will be convenient to consider operators

$$\gamma(\epsilon) = \prod_{v \in V} \gamma(v)^{(\epsilon, v)}, \qquad \text{for } \epsilon \in C_0, \tag{3.11a}$$

$$\mathfrak{s}(\tau) = \prod_{e \in E} \mathfrak{s}(e)^{(e, \tau)}, \qquad \text{for } \tau \in C_1. \tag{3.11b}$$

The sign of $\mathfrak{s}(\tau)$ depends on a choice of orientation for each $e \in E$ and an ordering of E, which we fix for the purpose of the proof. These operators satisfy $\gamma(\epsilon)|\epsilon'\rangle = (-1)^{(\epsilon, \epsilon')}|\epsilon'\rangle$ and $\mathfrak{s}(\tau)|\epsilon\rangle = (-1)^{\chi(\tau, \epsilon)}|\epsilon + \partial\tau\rangle$ for some function $\chi : C_1 \times C_0 \to \mathbb{Z}_2$, which depends on the arbitrary choices made.

Using relations (3.9) only, any monomial in the generators $\mathfrak{s}(e)$ and $\gamma(v)$ may be rewritten (perhaps up to a sign) as a product $\gamma(\epsilon)\mathfrak{s}(\tau)$ for some $\epsilon \in C_0$ and $\tau \in C_1$.

Now let r be a section of $\partial : C_1 \to B_0$, i.e. a linear map $B_0 \to C_1$ such that $\partial r = 1_{B_0}$. Notice that such r is guaranteed to exist, because ∂ is a linear map between vector spaces with image B_0. However, it is by no means unique.

For any $\tau \in C_1$ let $z(\tau) = \tau - r\partial\tau \in C_1$. Then we have $\tau = r\partial\tau + z(\tau)$ and $\partial z(\tau) = 0$, so $\mathfrak{s}(\tau)$ coincides with $\mathfrak{s}(r\partial\tau)$, possibly up to a sign. This means that, up to a sign, monomial $\gamma(\epsilon)\mathfrak{s}(\tau)$ depends on τ only through $\partial\tau$.

Using relations described so far, any relation in $\mathcal{A}_0$ may be reduced to

$$\sum_{\epsilon \in C_0} \sum_{\epsilon' \in B_0} c_{\epsilon, \epsilon'} \gamma(\epsilon)\mathfrak{s}(r\epsilon') = 0, \tag{3.12}$$

where $c_{\epsilon, \epsilon'}$ are complex coefficients.

[3] More precisely, $\mathcal{A}_0$ is isomorphic to a quotient of the free algebra on letters $\gamma(v)$, $\mathfrak{s}(e)$ by some two-sided ideal $\mathcal{I}$. We will describe a set of generators of $\mathcal{I}$.

Acting with the operator on the left hand side on the vector $|\epsilon''\rangle$ we obtain

$$\sum_{\epsilon\in C_0}\sum_{\epsilon'\in B_0} c_{\epsilon,\epsilon'}(-1)^{(\epsilon,\epsilon'+\epsilon'')}(-1)^{\chi(r\epsilon',\epsilon'')}|\epsilon''+\epsilon'\rangle = 0. \tag{3.13}$$

Since the set $\{|\epsilon''+\epsilon'\rangle\}_{\epsilon'\in B_0}$ is linearly independent in $\mathcal{F}$, each term of the summation over ϵ' vanishes separately. Therefore we have

$$\sum_{\epsilon\in C_0} c_{\epsilon,\epsilon'}(-1)^{(\epsilon,\epsilon'+\epsilon'')} = 0. \tag{3.14}$$

Now let $\epsilon_1 = \epsilon' + \epsilon''$, take any $\epsilon_2 \in C_0$ and multiply this equation by $(-1)^{(\epsilon_1,\epsilon_2)}$. Summing over all ϵ_1 and using the identity $\sum\limits_{\epsilon_1\in C_0}(-1)^{(\epsilon_1,\epsilon+\epsilon_2)} = 2^{|V|}\delta_{\epsilon,\epsilon_2}$ we get

$$c_{\epsilon_2,\epsilon'} = 0. \tag{3.15}$$

Since ϵ_2 and ϵ' were arbitrary, all coefficients c vanish. We have shown that any relation in $\mathcal{A}_0$ follows already from (3.9) and (3.10), which completes the proof.

4 Γ model

We will now construct a bosonic model equivalent to the fermionic one discussed in the previous section. Relations (3.9) will be satisfied as operator equations, but (3.10) will be imposed as a constraint on physical states. Due to the presence of Γ matrices in its formulation, we will refer to it as the Γ model [15]. Generators of the algebra $\mathcal{A}_0$ will be constructed as simple, local expressions in fields of the Γ model. Afterwards, we propose a correspondence between the Γ model and a certain $\mathbb{Z}_2$ gauge theory. The section is closed with a discussion of the Γ model and its constraints in case of toroidal geometries.

4.1 Definition of the model

In this section we will assume that the graph $\mathfrak{G}$ is such that every vertex has even degree. To a vertex v we associate the Clifford algebra with generators $\{\Gamma_*(v)\}\cup\{\Gamma(v,e)\}_{e\in\mathrm{St}(v)}$. Each generator squares to identity and anticommutes with every other generator located on the same vertex, but generators on different vertices commute. Clifford algebras associated to distinct vertices may be non-isomorphic, because we do not assume that all $v\in V$ have the same degree. Secondly, we construct an irreducible representation of the algebra associated to each vertex. There is some arbitrariness here, because there exist two non-isomorphic simple modules, corresponding to two possible values of $\Gamma_*(v)\prod\limits_{e\in\mathrm{St}(v)}\Gamma(v,e)$. For now we make some choice for every vertex. We will discuss its significance in subsection 4.2. Hilbert space $\mathcal{H}$ of the Γ model is defined as the tensor product of Hilbert spaces associated to individual vertices. Thus operators on distinct vertices commute. In this sense Γ model is bosonic.

Kinetic operators of the Γ model are defined in the following way. For every edge e we choose an orientation and put

$$S(e) = -i\Gamma(s(e),e)\Gamma(t(e),e). \tag{4.1}$$

For the opposite orientation we define $S(\overline{e}) := -S(e)$.

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A simple calculation shows that the map

$$\gamma(v) \mapsto \Gamma_*(v), \qquad \mathfrak{s}(e) \mapsto S(e) \tag{4.2}$$

is compatible with (3.9). However (3.10) does not hold as an operator relation. Nevertheless, if $\ell = (e_1, \dots, e_n)$ is a circuit, then $S(\ell) := S(e_1) \cdot \ldots \cdot S(e_n)$ is unitary, squares to identity and commutes with all $\Gamma_*(v)$ and $S(e)$. Therefore the subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ of all vectors ψ satisfying the constraint

$$S(\ell)\psi = \psi \qquad \text{for every circuit } \ell \tag{4.3}$$

is a representation of the algebra $\mathcal{A}_0$.

We claim that $\mathcal{H}_0$ is isomorphic (as a representation of $\mathcal{A}_0$) to a half of the Fock space, i.e. $\mathcal{H}_0 \cong \mathcal{F}_\alpha$ for some α. Remainder of this subsection is devoted to the proof of this fact.

Let $\ell = \left(e_1, \dots, e_{|E|}\right)$ be an Eulerian circuit. Then $S(\ell) = (-1)^\alpha \prod_{v \in V} \Gamma_*(v)$ for some α. Therefore acting with $S(\ell)$ on $\psi_0 \in \mathcal{H}_0$ we obtain

$$\left(\prod_{v \in V} \Gamma_*(v)\right) \psi_0 = (-1)^\alpha \psi_0. \tag{4.4}$$

This means that $\mathcal{H}_0$ is a direct sum of some number of copies of $\mathcal{F}_\alpha$. To show that the multiplicity is equal to one it is sufficient to demonstrate that $\dim(\mathcal{H}_0) = 2^{|V|-1}$. For this purpose let us first note that

$$\dim(\mathcal{H}) = \prod_{v \in V} 2^{\frac{\deg(v)}{2}} = 2^{|E|}. \tag{4.5}$$

Secondly, for every $[A] \in Z_1^*$ let $\mathcal{H}_{[A]}$ be the set of vectors ψ such that

$$S(\ell)\psi = (-1)^{([A],[\ell])}\psi \qquad \text{for every circuit } \ell. \tag{4.6}$$

We have a decomposition $\mathcal{H} = \bigoplus_{[A] \in Z_1^*} \mathcal{H}_{[A]}$. By the formula (2.1) there are $2^{|E|-|V|+1}$ summands, so the problem is reduced to checking that each $\mathcal{H}_{[A]}$ has the same dimension. This is achieved by considering the unitary operators

$$O(\tau) = \prod_{e \in E} \Gamma(s(e), e)^{(\tau, e)} \qquad \text{for } \tau \in C^1, \tag{4.7}$$

in which we choose orientation of each $e \in E$. Simple calculation shows that they satisfy

$$O(\tau)S(\ell) = (-1)^{([\tau],[\ell])} S(\ell) O(\tau) \qquad \text{if } \ell \text{ is a circuit.} \tag{4.8}$$

This implies that $O(\tau)\mathcal{H}_{[A]} \subseteq \mathcal{H}_{[A+\tau]}$, from which the result follows.

4.2 Choice of a representation

Recall that in the construction of our model it was necessary to choose a representation of the Clifford algebra at every vertex v. This is equivalent to specifying a relation between the action of $\Gamma_*(v)$ and $\prod_{e\in \mathrm{St}(v)} \Gamma(v,e)$. The two operators are proportional in every irreducible representation, but there are two possible values of the proportionality factor. One can resolve the ambiguity as follows. Let us choose some ordering of the set $\mathrm{St}(v)$. Then we may denote its elements as $e_1,\dots,e_{2n}$ with $n=\frac{\deg(v)}{2}$. Having done that we put

$$\Gamma_*(v) := i^n \Gamma(v,e_1)\cdot \ldots \cdot \Gamma(v,e_{2n}). \tag{4.9}$$

This is a consistent definition — element $\Gamma_*(v)$ anticommutes with all $\Gamma(v,e)$ and squares to 1. It is invariant with respect to even permutations of the indexing set $\{1,\dots,n\}$, but it changes sign under any odd permutation.

We see that our model is completely specified once we choose an ordering (modulo even permutations) of the set $\mathrm{St}(v)$ for each vertex v. We are not aware of a natural way to make this choice, save for the case of some very symmetric geometries. Thus it is crucial to understand its consequences. Any other construction of Γ_* is related to the chosen one by

$$\Gamma'_*(v) = (-1)^{(\eta,v)}\Gamma_*(v) \tag{4.10}$$

with some $\eta \in C_0$. Thus the space of distinct choices is affine over C_0. It does not seem to have a distinguished origin.

Now let us consider the unitary operators

$$T(\theta) = \left[\prod_{v\in V} \Gamma_*(v)^{(\partial\theta,v)}\right]\cdot\left[\prod_{e\in E} S(e)^{(\theta,e)}\right] \qquad \text{for } \theta\in C_1, \tag{4.11}$$

whose signs depend on a choice of orientations of edges and an ordering of E. They commute with all $S(e)$ and satisfy

$$T(\theta)\Gamma_*(v)T(\theta)^{-1} = (-1)^{(\partial\theta,v)}\Gamma_*(v). \tag{4.12}$$

This establishes that constructions of our model related by (4.10) are unitarily equivalent if $\eta=\partial\theta$. Thus they describe the same physics for any choice of hamiltonian built of fermionic parity and kinetic operators. Identifying equivalent models we see that the set of distinct versions of the Γ model is affine over the homology group $C_0/B_0\cong\mathbb{Z}_2$, or in simpler words — it has two elements. They correspond to two possible values of α in (4.4). Indeed, redefinition (4.10) with η representing a nonzero homology class (i.e. a sum of an odd number of vertices) changes the sign of the operator $\prod_{v\in V}\Gamma_*(v)$ while keeping the form of constraints (4.3) invariant.

The discussion above may be phrased in the language of higher symmetries [39] as follows: construction of our model has a sort of gauge freedom, with gauge transformations parametrized by 1-chains. If the graph $\mathfrak{G}$ is the one-skeleton of a closed d-dimensional manifold[4] X, there is a Poincaré-duality between 1-chains and $(d-1)$-cochains. In this

[4] X does not have to be orientable, because we need only Poincaré duality over $\mathbb{Z}_2$.

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sense Γ model has a $(d-1)$-form $\mathbb{Z}_2$ gauge invariance. We may identify $\prod_{v\in V}\Gamma_*(v)$ as the unique nontrivial gauge-invariant d-holonomy operator. Choice of a particular representation involves fixing the gauge as well as the value of this operator.

We will now describe how to construct data needed to completely determine the Γ model corresponding to a prescribed value of α. It suffices to do this for $\alpha = 1$, the other case being obtained by a transformation (4.10) with any $\eta \notin B_0$. Let $\ell = \left(e_1, \ldots, e_{|E|}\right)$ be an Eulerian circuit. For every $v \in V$ there are exactly $n := \frac{\deg(v)}{2}$ indices $1 \leq j_1 < j_2 < \ldots < j_n \leq |E|$ such that $s(e_{j_i}) = v$. We define an ordering on $\mathrm{St}(v)$ by

$$e_{j_1-1} < e_{j_1} < e_{j_2-1} < e_{j_2} < \ldots < e_{j_n-1} < e_{j_n}, \tag{4.13}$$

where $e_0 := e_{|E|}$. It is easy to check that then $S(\ell) = -\Gamma_*(v)$, so $\alpha = 1$. In particular, distinct choices of the Eulerian circuit ℓ give rise to orderings which are equivalent in the sense described in the previous paragraph.

4.3 Modified constraints and $\mathbb{Z}_2$ gauge fields

Consider coupling fermions to an external lattice $\mathbb{Z}_2$ gauge field. The gauge field is a cochain $A \in C^1$ subject to gauge transformations $A \mapsto A + \delta\theta$ with $\theta \in C^0$. Thus gauge orbits are parametrized by equivalence classes $[A] \in C^1/B^1 = Z_1^*$. The minimal coupling rule asserts that each occurence of $\mathfrak{s}(e)$ in the fermionic hamiltonian should be replaced by $\mathfrak{s}_A(e) := \mathfrak{s}(e) \cdot (-1)^{(A,e)}$. These operators satisfy the same relations as the original $\mathfrak{s}(e)$ except of (3.10), which is replaced by

$$\mathfrak{s}_A(e_1) \cdot \ldots \cdot \mathfrak{s}_A(e_n) = (-1)^{([A],[\ell])} \qquad \text{for every circuit } \ell = (e_1, \ldots, e_n). \tag{4.14}$$

Now consider the bosonization map

$$\gamma(v) \mapsto \Gamma_*(v), \qquad \mathfrak{s}_A(e) \mapsto S(e). \tag{4.15}$$

In order for this prescription to be compatible with the relation (4.14) it is necessary to restrict attention to the subspace $\mathcal{H}_{[A]} \subseteq \mathcal{H}$ of vectors ψ satisfying the constraint (4.6). Notice that the form of this condition is gauge-independent, because $(A, [\ell])$ depends only on the gauge orbit $[A]$ of A for every circuit ℓ. On the other hand, the form of the bosonization map (4.15) does depend on the choice of gauge.

We conclude that in order to couple fermions to a $\mathbb{Z}_2$ gauge field it is sufficient to change the form of constraint to (4.6), without changing the form of hamiltonian expressed in terms of $\gamma(v)$ and $S(e)$ operators. It remains to describe the structure of the $\mathcal{A}_0$-module $\mathcal{H}_{[A]}$. We pick an Eulerian circuit ℓ and an element $\psi \in \mathcal{H}_{[A]}$. Then

$$(-1)^\alpha \left(\prod_{v\in V}\Gamma_*(v)\right)\psi = S(\ell)\psi = (-1)^{([A],\zeta)}\psi, \tag{4.16}$$

where we used the fact that $[\ell] = \zeta$ for any Eulerian circuit ℓ.

We conclude that $\mathcal{H}_{[A]}$ is isomorphic to a direct sum of some number of copies of $\mathcal{F}_{\alpha+([A],\zeta)}$. Since $\dim(\mathcal{H}_{[A]}) = \dim(\mathcal{H}_0)$, the multiplicity is equal to one.

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We have shown that the full Hilbert space of the Γ model decomposes as a direct sum of subspaces describing fermions coupled to all possible external $\mathbb{Z}_2$ gauge fields. Interestingly, the allowed value of fermionic parity depends on the "magnetic" observable $([A], \zeta)$.

One could ask whether it is possible to promote the gauge field to a dynamical degree of freedom. In order to write down a kinetic term for the A field it would be necessary to invoke "electric" operators which connect subspaces corresponding to different values of the gauge field. Before answering to what extent such operators exist in our model, we briefly review the construction of the conventional $\mathbb{Z}_2$ gauge theory [40, 41] coupled to fermions.

The Hilbert space is defined initially as the tensor product of the fermionic Hilbert space and the Hilbert space for gauge fields. The latter has an orthonormal basis $\{|A\rangle\}$ with A running over all elements of C^1. Magnetic operators $U(\tau)$ are parametrized by chains $\tau \in C_1$. They act on basis states according to the formula $U(\tau)|A\rangle = (-1)^{(A,\tau)}|A\rangle$. Electric operators $W(\omega)$ are parametrized by $\omega \in C^1$ and defined by $W(\omega)|A\rangle = |A+\omega\rangle$. Thus one has braiding relations $U(\tau)W(\omega) = (-1)^{(\omega,\tau)}W(\omega)U(\tau)$.

In the next step one introduces Gauss' operators $G(\theta) = \gamma(\theta)W(\delta\theta)$ for $\theta \in C_0$. They implement $\mathbb{Z}_2$ gauge transformations. Only gauge-invariant states ($G(\theta)\psi = \psi$) are regarded as physical. This defines the true Hilbert space of the theory. Taking $\theta = \sum\limits_{v \in V} v$ one finds that all physical states are eigenvectors of γ to eigenvalue one, so there are no states with odd number of fermions.

The algebra of gauge-invariant operators ($G(\theta)OG(\theta)^{-1} = O$) is generated by dressed kinetic operators $\mathfrak{s}_g(e) = \mathfrak{s}(e) \cdot U(e)$ and electric operators $W(e)$. There are magnetic observables $U(\tau)$ for $\partial\tau = 0$, but these may be expressed in terms of kinetic operators. Indeed, for ℓ being a circuit

$$U([\ell]) = \mathfrak{s}_g(\ell). \tag{4.17}$$

Similarly the charge operators may be expressed[5] in terms of electric operators:

$$\gamma(v) = W(\delta v). \tag{4.18}$$

The only independent relations between our chosen generators are (3.9b) with $\mathfrak{s}$ replaced by $\mathfrak{s}_g$, the following properties of W:

$$W(\omega) = W(\omega)^* = W(\omega)^{-1}, \qquad W(\omega_1 + \omega_2) = W(\omega_1)W(\omega_2), \tag{4.19}$$

and braiding relations between kinetic and electric operators

$$\mathfrak{s}_g(e)W(\omega) = (-1)^{(\omega,e)}W(\omega)\mathfrak{s}_g(e). \tag{4.20}$$

Now we return to the Γ model considered without any constraints on physical states. We ask if the algebra of gauge-invariant operators of $\mathbb{Z}_2$ gauge theory may be represented on its Hilbert space. We would like to map $\mathfrak{s}_g(e)$ to $S(e)$ and $\gamma(v)$ to $\Gamma_*(v)$. This is consistent with local relations in gauge theory, but it is inconsistent with the global relation $\prod\limits_{v \in V} \gamma(v) = 1$, since we have instead $\prod\limits_{v \in V} \Gamma_*(v) = (-1)^{\alpha} S(\ell)$ for an Eulerian circuit ℓ.

[5] We regard gauge-invariant operators as acting on the physical Hilbert space only, so identities which follow from the Gauss' law are written as operator relations.

On the gauge theory side the problematic relation is a consequence of the Gauss' law, so we would like to interpret the Γ model as a gauge theory with deformed Gauss' law. Such deformation has the consequence that it is not possible to represent operators $W(e)$ in a way compatible with $W(e)W(e') = W(e')W(e)$ and braiding relations (4.20), because these operators would have to anticommute with the c-number $\Gamma_*(v)S(\ell) = (-1)^\alpha$, which is absurd.[6] This argument does not concern operators $W(\omega)$ with ω orthogonal to ζ, i.e. those ω which are sums of even numbers of edges. To construct a convenient basis of C_1^{even}, the orthogonal complement of ζ, let $\ell = (e_1, \dots, e_{|E|})$ be an Eulerian circuit. Put $\epsilon_i = e_{i-1} + e_i \in C_1$ for $2 \leq i \leq |E|$. Then ϵ_i form a basis of C_1^{even} and have the convenient property that each ϵ_i is a sum of two edges which meet at the vertex $v_i := s(e_i)$. Since each edge $e \in E$ is equal to e_i for exactly one i, this construction defines a partition of each set $\mathrm{St}(v)$ into a disjoint union of $\frac{\deg(v)}{2}$ pairs of the form e_{i-1}, e_i (where $e_0 := e_{|E|}$) with $1 \leq i \leq |E|$ such that $v = v_i$. Now define

$$\mathcal{W}(\epsilon_i) = (-1)^{\kappa_i} \cdot i\Gamma(v_i, e_{i-1})\Gamma(v_i, e_i) \qquad \text{for } 2 \leq i \leq |E|, \tag{4.21}$$

where $\kappa_i \in \mathbb{Z}_2$ is not yet specified. Operators $\mathcal{W}(\epsilon_i)$ are our candidates for representatives of $W(\epsilon_i)$. We have $\mathcal{W}(\epsilon_i)\mathcal{W}(\epsilon_j) = \mathcal{W}(\epsilon_j)\mathcal{W}(\epsilon_i)$ and $\mathcal{W}(\epsilon_i)^2 = 1$, so we may well-define $\mathcal{W}(\omega)$ for any $\omega \in C_1^{\mathrm{even}}$ by demanding that $\mathcal{W}(\omega_1 + \omega_2) = \mathcal{W}(\omega_1)\mathcal{W}(\omega_2)$. For example

$$\mathcal{W}(e_1 + e_n) = \prod_{i=2}^{|E|} \mathcal{W}(\epsilon_i), \tag{4.22}$$

since $e_1 + e_n = \sum_{i=2}^{|E|} \epsilon_i$. With this definition relations (4.19) and (4.20) are satisfied. Furthermore, we can choose κ_i in such a way that $\mathcal{W}(\delta v) = \Gamma_*(v)$ is satisfied for every vertex other than $v_1 := s(e_1) = t(e_n)$. For example if elements $\Gamma_*(v)$ are constructed as in the discussion surrounding equation (4.13), one may take all $\kappa_i = 0$. In any case we have

$$\mathcal{W}(\delta v_1) = \mathcal{W}\left(\sum_{v \neq v_1} \delta v\right) = \prod_{v \neq v_1} \Gamma_*(v) = (-1)^\alpha S(\ell) \cdot \Gamma_*(v_1). \tag{4.23}$$

This means that for the single vertex v_1 the Gauss' law is modified by the factor $(-1)^\alpha S(\ell)$.

We are now ready to define the gauge theory corresponding to the Γ model with no constraints imposed. Elementary fermionic operators as well as U and W operators are constructed as in the conventional gauge theory. The only modification is in the definition of the Gauss' operators, which are taken to be

$$G(v) = \begin{cases} \gamma(v)W(\delta v) & \text{for } v \neq v_1, \\ (-1)^\alpha \gamma(v)U(\zeta)W(\delta v) & \text{for } v = v_1. \end{cases} \tag{4.24}$$

[6] One way to avoid this conclusion is to consider the direct sum of Hilbert spaces of two versions of the Γ model corresponding to two values of α. Then $(-1)^\alpha$ is promoted to an operator with eigenvalues ± 1, so it is possible to introduce operators which anticommute with it. Such construction was considered in [28], but this is not what we would like to do here.

This has the consequence that also the algebra of gauge invariant operators is modified. We study properties of this gauge theory and its generalizations in section 5. Here we summarize those results obtained there which are directly relevant for the correspondence with the Γ model:

- An isomorphism between the algebra of gauge-invariant operators in gauge theory and $\mathrm{End}(\mathcal{H})$ is constructed. Operators constructed of even numbers of fermions and Wilson lines are mapped to local operators in the Γ model, but electric operators are represented in a way which is non-local and depends on the choice of an Eulerian circuit. Similarly, there exist non-local operators in gauge theory corresponding to $\Gamma(v, e)$ from the Γ model.

- The definition of Gauss' operators suggests that there is an inherent non-locality and lack of symmetry between distinct vertices in the proposed gauge theory. We demonstrate that under certain assumptions about the underlying geometry these pathologies can be healed by a canonical transformation.

- The Gauss' law, which is imposed as a constraint in the gauge theory picture, holds identically in the Γ model. Therefore all states and all operators in the Γ model are gauge invariant.

- Relation between the total number of fermions mod 2 and the value of $[A]$ satisfied in the Γ model is a consequence of the Gauss' law on the gauge theory side.

It is interesting to interpret the algebra of $\{\Gamma(v, e)\}$, the elementary fields of the Γ model, in terms of quantum numbers defined in gauge-theoretical language. To this end we inspect the braiding relations

$$\Gamma_*(v')\Gamma(v, e) = (-1)^{(v,v')}\Gamma(v, e)\Gamma_*(v'), \tag{4.25a}$$

$$S(\ell)\Gamma(v, e) = (-1)^{([\ell],e)}\Gamma(v, e)S(\ell) \qquad \text{if } \ell \text{ is a circuit.} \tag{4.25b}$$

The first relation asserts that $\Gamma(v, e)$ flips the value of fermionic parity at the vertex v, i.e. it creates or annihilates a fermion. The second one means that action of $\Gamma(v, e)$ changes the value of the holonomy along any loop which contains the edge e. There is no operator that creates or annihilates a single fermion without disturbing the values of holonomies or a one that acts as an electric field operator on a single edge without creating any fermions, because that would contradict the relation

$$\text{Total number of fermions} \pmod 2 \; = \; \alpha + ([A], \zeta). \tag{4.26}$$

The preceding discussion justifies thinking of $\Gamma(v, e)$ as a composite of a fermion and a lump of electromagnetic field, as in the so-called flux attachment mechanism.

According to the presented picture, the role of constraints (4.3) present in our bosonization map is to get rid of the electromagnetic degrees of freedom present in the Γ model. We close this discussion with the remark that constraints can be divided into two classes:

1. Constraints which correspond to homologically trivial loops, i.e. circuits ℓ such that the cycle $[\ell]$ belongs to B_1. It is sufficient to impose one such constraint for every face of the lattice. Constraints of this type are local, and hence can be implemented by introducing in the hamiltonian local terms which penalize their violation. They reduce the Hilbert space from $\mathcal{H}$ to the direct sum of subspaces corresponding to gauge orbits of flat gauge fields, i.e. to $\bigoplus_{[A]\in H^1} \mathcal{H}_{[A]}$.

2. Constraints which correspond to loops of nonzero homology class. Once constraints of the first type are imposed, operators corresponding to distinct representatives of the same homology class become equivalent. It is sufficient to impose one such constraint for every element of some basis of H_1. This chooses from the set of all flat gauge fields the trivial gauge field $A = 0$.

4.4 Example: toroidal geometries

In this subsection we construct the Γ model on a torus with $L_1 \times \ldots \times L_d$ lattice sites, with each $L_i \geq 3$. In the case of $d = 2$ and even L_i we present a full solution of constraints (4.3).

Lattice vertices are labeled by d-tuples of integers, with two d-tuples identified if they differ by a tuple whose i-th entry is a multiple of L_i for each i. Sets of edges and faces are the obvious ones. Clearly every vertex has even degree.

Operator $\Gamma(v, e)$ with edge e in positive or negative i-th direction is denoted by $\Gamma_{\pm i}(v)$. Furthermore, we introduce

$$\Gamma_*(v) = (-1)^{(\eta,v)} \cdot i^d \prod_{i=1}^{d} \Gamma_i(v)\Gamma_{-i}(v) \tag{4.27}$$

where η is a 0-chain. With this convention

$$\alpha = \sum_{v\in V} (\eta, v) + \sum_{i=1}^{d} \prod_{j\neq i} L_j, \tag{4.28}$$

as can be easily evaluated by computing the product $\prod_{\ell} S(\ell)$ with ℓ running through the set of all straight lines winding once around the torus.

Let f be a face lying in the plane spanned by directions $1 \leq i < j \leq d$, with vertices A, B, C, D ordered counterclockwise, starting from the south-west corner (see figure 1).

The constraint (4.3) for the circuit around the boundary of f is of the form

$$\mathcal{P}(f)|\text{phys}\rangle = |\text{phys}\rangle, \tag{4.29a}$$

$$\mathcal{P}(f) = -\Gamma_{i,j}(A)\Gamma_{j,-i}(B)\Gamma_{-i,-j}(C)\Gamma_{-j,i}(D), \tag{4.29b}$$

where $\Gamma_{k,l}(v) := \Gamma_k(v)\Gamma_l(v)$. We note the mnemonic rule that in the above, indices $\pm i, \pm j$ labeling gamma matrices are arranged in a cycle.

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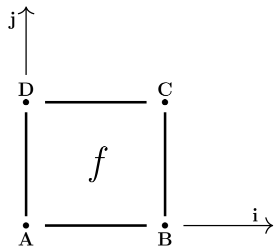

Figure 1. Labels of vertices for a face f lying in the plane spanned by directions i, j.

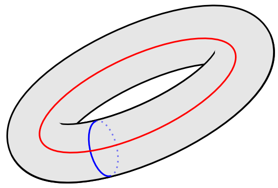

Figure 2. Two loops wrapping the 2-dimensional torus.

The only other constraints correspond to d independent loops wrapping around the whole torus (see figure 2). They take the form

$$\mathcal{L}_j(v)|\mathrm{phys}\rangle = |\mathrm{phys}\rangle, \qquad j = 1, \ldots, d, \tag{4.30a}$$

$$\mathcal{L}_j(v) := -i^{L_j} \prod_{k=0}^{L_j-1} \Gamma_{j,-j}(t_j^k \cdot v), \tag{4.30b}$$

where $v \in V$ is a reference vertex and t_i is the transformation of V defined by

$$t_i \cdot (v_1, \ldots, v_d) = (v_1, \ldots, v_i + 1, \ldots, v_d). \tag{4.31}$$

We note that $\mathcal{L}_i$ are unitary, hermitian and commute with each other.

We now confine ourselves to the case of $d = 2$ and all L_i even. Consider the operators

$$\Xi_1(v) = \prod_{k=0}^{L_2-1} \Gamma_{1,(-1)^k 2}(t_2^k \cdot v), \tag{4.32a}$$

$$\Xi_2(v) = \prod_{k=0}^{L_1-1} \Gamma_{(-1)^k 1,2}(t_1^k \cdot v). \tag{4.32b}$$

They are unitary, hermitian and commute with all $\mathcal{P}(f)$, $\Gamma_*(v)$ and with each other. Moreover, they flip the values of corresponding $\mathcal{L}_j$:

$$\Xi_i(v)\mathcal{L}_j(v) = (-1)^{\delta_{i,j}} \mathcal{L}_j(v)\Xi_i(v). \tag{4.33}$$

This means that pairs $\{\mathcal{L}_1(v), \Xi_1(v)\}$ and $\{\mathcal{L}_2(v), \Xi_2(v)\}$ generate two independent copies of the Pauli algebra. Thus solutions of plaquette constraints are organized in quadruplets, each of which contains precisely one solution of the loop constraint (4.30a). Given any state in such a quadruplet, the desired state satisfying (4.30a) may be easily obtained by acting with an appropriate element of the algebra generated by $\mathcal{L}_i$ and Ξ_i.

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We remark that similar trick can be applied for other geometries, including higher dimensions, provided that the cycle ζ is a boundary. The role of Ξ_i is played by electric operators $\mathcal{W}(\tau)$ with $\delta\tau = 0$. These exist because $(\tau, \zeta) = 0$ for $\tau \in Z^1$, $\zeta \in B_1$.

Having dealt with the loop constraints, we proceed to the analysis of plaquettes. It will be convenient to divide the lattice into two complementary alternating sublattices, called even and odd. For example we may declare vertex $v = (v_1, v_2)$ to be even if $v_1 + v_2 = 0 \pmod 2$. Parity of a face f is defined as the parity of its south-west corner.

We will construct solutions of constraints which are simultaneous eigenvectors of $\Gamma_*(v)$ to eigenvalues $(-1)^{(\eta,v)}$. Solutions with other eigenvalues may then be obtained by acting with kinetic operators, which commute with all constraints. After this restriction, we have the relation $\Gamma_{1,-1}(v)\Gamma_{2,-2}(v) = -1$ for every vertex v. This can be used to simplify the plaquette constraints to the form

$$\mathcal{P}(f) = \Gamma_{1,2}(A)\Gamma_{1,2}(C)\Gamma_{1,-2}(B)\Gamma_{1,-2}(D). \tag{4.34}$$

Now we introduce new local operators by the formulas

$$\sigma_3(v) = \begin{cases} i\Gamma_{1,2}(v) & \text{for } v \text{ even}, \\ i\Gamma_{1,-2}(v) & \text{for } v \text{ odd}, \end{cases} \qquad \sigma_1(v) = \begin{cases} -i\Gamma_{1,-2}(v) & \text{for } v \text{ even}, \\ i\Gamma_{1,2}(v) & \text{for } v \text{ odd}. \end{cases} \tag{4.35}$$

Then with the definition $\sigma_2(v) = -i\sigma_3(v)\sigma_1(v)$ we have

$$\sigma_2(v) = i\Gamma_{1,-1}(v) \qquad \text{for every } v \in V. \tag{4.36}$$

One can check that for each v operators $\{\sigma_i(v)\}_{i=1}^3$ satisfy the standard relations obeyed by Pauli matrices, which justifies the chosen notation.

In terms of the new variables, plaquette operators take the form

$$\mathcal{P}(f) = \begin{cases} \prod\limits_{v\in\{A,B,C,D\}} \sigma_3(v) & \text{for } f \text{ even}, \\ \prod\limits_{v\in\{A,B,C,D\}} \sigma_1(v) & \text{for } f \text{ odd}. \end{cases} \tag{4.37}$$

In this form plaquette constraints are readily recognized as equations defining ground states of the famous Kitaev's toric code [11]. It is well-known that there exist four solutions, corresponding to two values of $\mathcal{L}_1$ and $\mathcal{L}_2$. This is also in accord with our general finding about the Γ model. For completeness we provide a prescription to construct these states in the next paragraph.

We work in the standard eigenbasis of $\sigma_3(v)$ operators, so our basis states are labeled by elements $\omega \in C_0$ and satisfy

$$\sigma_3(v)|\omega\rangle = (-1)^{(\omega,v)}|\omega\rangle, \tag{4.38a}$$

$$\sigma_1(v)|\omega\rangle = |\omega + v\rangle. \tag{4.38b}$$

In order to have $\mathcal{P}(f)|\omega\rangle = |\omega\rangle$ for even faces f, we need to have

$$(\omega, A + B + C + D) = 0, \tag{4.39}$$

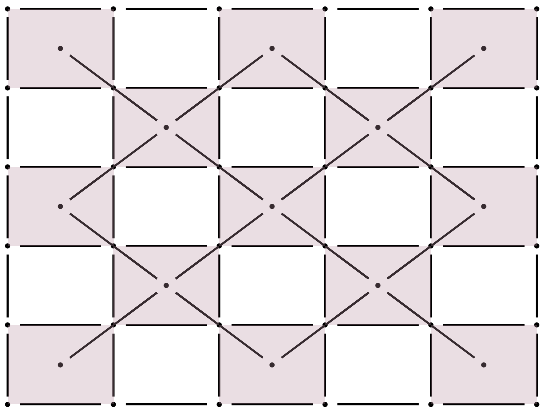

Figure 3. Lattice whose vertices are the centres of even (shaded) faces of the original lattice. Its edges and faces correspond to vertices and odd (white) faces of the original lattice, respectively.

where A, B, C, D are the four vertices of any even face. Every such chain ω will be called admissible. Geometrically this condition means that ω may be identified with a 1-cocycle on the lattice whose vertices are the even faces of the orignal lattice (see figure 3).

Calculation analogous to the proof of (2.1) shows that there exist $\frac{L_1 L_2}{2} + 1$ admissible chains. Now consider the state

$$|\mathrm{ref}\rangle = 2^{-\frac{L_1 L_2 + 2}{4}} \sum_{\omega \text{ admissible}} |\omega\rangle. \tag{4.40}$$

Clearly we have $\mathcal{P}(f)|\mathrm{ref}\rangle = |\mathrm{ref}\rangle$ for every face f and $\langle \mathrm{ref}|\mathrm{ref}\rangle = 1$.

State $|\mathrm{ref}\rangle$ satisfies all plaquette constraints, but does not satisfy the loop constraints. In this paragraph we solve this difficulty. As a fist step towards this goal, we express $\mathcal{L}$ and Ξ operators in terms of Pauli matrices. We take the reference vertex v to be even. Then

$$\mathcal{L}_1(v) = -\prod_{k=0}^{L_1-1} \sigma_2(t_1^k \cdot v), \qquad \Xi_1(v) = (-1)^{\frac{L_2}{2}} \prod_{k=0}^{L_2-1} \sigma_3(t_2^k \cdot v), \tag{4.41a}$$

$$\mathcal{L}_2(v) = -\prod_{k=0}^{L_2-1} \sigma_2(t_2^k \cdot v), \qquad \Xi_2(v) = (-1)^{\frac{L_1}{2}} \prod_{k=0}^{L_1-1} \sigma_3(t_1^k \cdot v). \tag{4.41b}$$

Using the above and the definition of $|\mathrm{ref}\rangle$ we obtain eigenvalue equations

$$\mathcal{L}_1(v)\Xi_2(v)|\mathrm{ref}\rangle = \mathcal{L}_2(v)\Xi_1(v)|\mathrm{ref}\rangle = -|\mathrm{ref}\rangle. \tag{4.42}$$

This eigensystem combined with the relations obeyed by $\mathcal{L}$ and Ξ operators implies that projection of $|\mathrm{ref}\rangle$ onto the joint eigenspace of $\mathcal{L}_1$ and $\mathcal{L}_2$ to eigenvalue 1 has norm $\frac{1}{2}$. To obtain a properly normalized state, we multiply this projection by 2:

$$|0\rangle = 2 \cdot \frac{1 + \mathcal{L}_1(v)}{2} \frac{1 + \mathcal{L}_2(v)}{2} |\mathrm{ref}\rangle. \tag{4.43}$$

We close this section with a remark that the presented method of solving constraints can be generalized to all geometries such that there exists a partition of the set of faces

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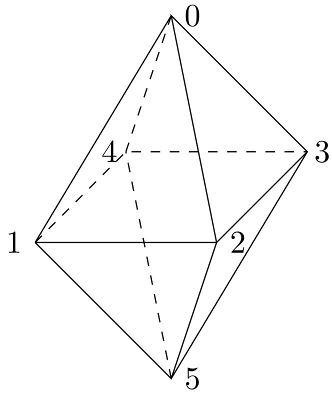

Figure 4. Octahedron.

(say, into "white" and "shaded" faces) such that no two faces of the same colour share an edge. Then one can construct a basis of solutions of "shaded" constraints consisting of products states, which are permuted by the action of "white" constraints. Thus the sum of all elements of this basis satisfies constraints of both types. One particularly simple decomposition of the two-sphere for which this can be carried out is the octahedron, see figure 4. Unfortunately, the relevant condition is never satisfied in the case of geometries of dimension higher than two. One can always obtain a solution of all constraints by acting with the projection operator $\prod_f \frac{1+\mathcal{P}(f)}{2}$ on some reference state, but this does not lead to a description as explicit as in (4.40) and (4.43).

4.5 Example: quadratic fermionic hamiltonians

Here we illustrate the bosonization procedure by applying it to hamiltonians of the form

$$H = \sum_{e \in E_{\rm or}} h_e \ \phi(s(e))\phi(t(e))^* + \sum_{v \in V} \nu_v \ \phi(v)^* \phi(v), \tag{4.44}$$

where $h_{\overline{e}} = \overline{h_e}$, while ν_v are real. This hamiltonian may be rewritten as

$$H = \sum_{e \in E_{\rm or}} h_e \ \frac{1+\gamma(s(e))}{2} \mathfrak{s}(e) \frac{1+\gamma(t(e))}{2} + \sum_{v \in V} \nu_v \ \frac{1-\gamma(v)}{2}, \tag{4.45}$$

from which we read off the bosonized form:

$$H_\Gamma = \sum_{e \in E_{\rm or}} h_e \ \frac{1+\Gamma_*(s(e))}{2} S(e) \frac{1+\Gamma_*(t(e))}{2} + \sum_{v \in V} \nu_v \ \frac{1-\Gamma_*(v)}{2}. \tag{4.46}$$

This hamiltonian commutes with $S(\ell)$ for every circuit ℓ. Thus it has a local symmetry generated by operators $\mathcal{P}(f)$, which are defined as $S(\ell)$ with $[\ell] = \partial f$, and further operators labeled by loops whose classes generate the homology group H_1.

We will now describe the spectrum of H_Γ. First, consider the one-particle subspace of the fermionic system. It is governed by the $|V| \times |V|$ matrix $\{\langle v'|H|v\rangle\}_{v,v' \in V}$. Denote its eigenvalues by $\lambda_i[h,\nu]$, $i = 1, \dots, |V|$. The eigenvalues of H are $\lambda_I[h,\nu] = \sum_{i \in I} \lambda_i[h,\nu]$, indexed by subsets I of $\{1,\dots,|V|\}$. Eigenvalues of H_Γ restricted to the subspace $\mathcal{H}_0$ are exactly $\lambda_I[h,\nu]$, with a restriction $|I| = \alpha \pmod 2$. To understand the spectrum of H_Γ acting on $\mathcal{H}_{[A]}$ with $[A] \neq 0$ notice that minimal coupling to a $\mathbb{Z}_2$ gauge field amounts to replacing h_e by $h_e^A = h_e \cdot (-1)^{(A,e)}$. Therefore the eigenvalues of H_Γ in $\mathcal{H}_{[A]}$ are $\lambda_I[h^A,\nu]$ with $|I| = \alpha + ([A],\zeta) \pmod 2$.

To enforce the plaquette constraints dynamically, consider adding to H_Γ the local term

$$H_c = J\sum_f \frac{1-\mathcal{P}(f)}{2}. \tag{4.47}$$

This leaves unchanged the eigenvalues $\lambda_I[h^A,\nu]$ for flat gauge fields A and increases every other eigenvalue by at least J. Thus for J large enough all low energy eigenstates correspond to flat gauge fields.

5 Deformed $\mathbb{Z}_2$ gauge theories

In this section we demonstrate that the gauge theory proposed in subsection 4.3 is indeed equivalent to the Γ model, even though the correspondence is local only for some operators. The proof relies on technical facts presented in the appendix A. Afterwards we present a certain generalization of this model, in which Gauss' operators of conventional $\mathbb{Z}_2$ gauge theory are modified by including phases depending on values of the holonomies. Similar mechanism is present in the Dijkgraaf-Witten theory and has been applied in the bosonization map introduced in [13, 14, 34]. In contrast to Dijkgraaf-Witten models, here we are not restricting attention to topological gauge theories.[7] Modified Gauss' operators are classified up to (in general non-local) canonical transformations. We use this result to show how the gauge theory corresponding to the Γ model can be formulated in a local way.

5.1 Gauge invariant operators

We will now describe the algebra of gauge invariant operators for Gauss' operators of the form (4.24) and explain how it is represented on the Γ model Hilbert space.

Operator built of $\{X(v),Y(v)\}_{v\in V}$ will be said to be of charge $q \in C_0$ and denoted by the generic symbol $\Upsilon(q)$ if it satisfies the braiding relation

$$\gamma(v)\Upsilon(q) = (-1)^{(q,v)}\Upsilon(q)\gamma(v). \tag{5.1}$$

Every operator may be written down as a linear combination of operators of the form $\mathcal{O} = U(\tau)W(\sigma)\Upsilon(q)$. All such operators are eigenvectors of the group of gauge transformations, so the most general gauge-invariant operator is a linear combination of operators of the form $\mathcal{O}$ with each term separately gauge invariant. We proceed to find conditions for gauge invariance of $\mathcal{O} \neq 0$. Its braiding with Gauss' operators is given by

$$G(v)\mathcal{O}G(v)^{-1} = (-1)^{(\partial\tau,v)+(v_1,v)(\zeta,\sigma)+(q,v)}\mathcal{O}, \tag{5.2}$$

[7] In other words, we are working with principal bundles over 1-skeleta which do not necessarily extend to the 2-skeleton of the underlying space. Secondly, considered models depend on a choice of an arbitrary 1-cycle. We would expect only 1-cycles dual to characteristic classes to appear in topological field theories.

so gauge invariance of $\mathcal{O}$ is equivalent to the equation

$$\partial\tau = q + (\zeta,\sigma)v_1. \tag{5.3}$$

Contracting this relation with $\sum\limits_{v\in V} v$ we infer $\sum\limits_{v\in V}(q,v) = (\zeta,\sigma)$. Thus there are two possibilities: q is a sum of an even or odd number of vertices.

In the former case $(\zeta,\sigma)=0$ (so σ is a sum of an even number of edges) and $\partial\tau = q$. Operator $\mathcal{O}$ of this type is a product of

(a) Wilson lines, which are allowed to terminate at charges in the usual way,

(b) $W(\sigma)$ with $(\zeta,\sigma)=0$.

These two factors of $\mathcal{O}$ are separately gauge invariant.

In the case that q contains an odd number of vertices, we need $(\zeta,\sigma)=1$ and hence $\partial\tau = q+v_1$. Thus $\mathcal{O}$ is a product of an operator of the former type and $X(v_0)W(e)$ with some edge e.

In order to construct a set of generators convenient for comparisons with the Γ model, choose an Eulerian circuit $\ell = (e_1,\dots,e_{|E|})$. We put $v_i = s(e_i)$ $(1\le i\le |E|)$, $\epsilon_i = e_{i-1}+e_i$ $(2\le i\le |E|)$ and $e_0 = e_{|E|}$. The algebra under consideration is generated by the set $\{\mathfrak{s}_g(e_i)\}_{i=1}^{|E|} \cup \{W(\epsilon_i)\}_{i=2}^{|E|} \cup \{K\}$, where $K = X(v_1)W(e_0)$. Operators $U(\tau)$ for $\tau\in Z_1$ can be expressed in terms of $\{\mathfrak{s}_g(e_i)\}$, while $\gamma(v)$ is, perhaps up to a sign or a factor $U(\zeta)$, the product of some number of $W(\epsilon_i)$. The following relations are satisfied:

$$-\mathfrak{s}_g(e_i) = \mathfrak{s}_g(e_i)^* = \mathfrak{s}_g(e_i)^{-1}, \qquad \mathfrak{s}_g(e_i)\mathfrak{s}_g(e_j) = (-1)^{(\partial e_i,\partial e_j)}\mathfrak{s}_g(e_j)\mathfrak{s}_g(e_i), \tag{5.4a}$$

$$W(\epsilon_i) = W(\epsilon_i)^* = W(\epsilon_i)^{-1}, \qquad W(\epsilon_i)W(\epsilon_j) = W(\epsilon_j)W(\epsilon_i), \tag{5.4b}$$

$$\mathfrak{s}_g(e_i)W(\epsilon_j) = (-1)^{(e_i,\epsilon_j)}W(\epsilon_j)\mathfrak{s}_g(e_i), \tag{5.4c}$$

$$K = K^* = K^{-1}, \tag{5.4d}$$

$$K\mathfrak{s}_g(e_i) = (-1)^{(e_i,e_0+\delta v_1)}\mathfrak{s}_g(e_i)K, \qquad KW(\epsilon_i) = W(\epsilon_i)K. \tag{5.4e}$$

We have already verified that the map

$$\mathfrak{s}_g(e_i)\mapsto S(e_i), \qquad W(\epsilon_i)\mapsto \mathcal{W}(e_i), \tag{5.5}$$

defined in subsection 4.3, preserves all relations above not involving K. Thus it remains only to propose a representative of K in the Γ model. One can choose simply

$$K\mapsto \Gamma(v_1,e_0), \tag{5.6}$$

which is consistent with relations (5.4).

We claim that the proposed map well-defines an isomorphism between the algebra of gauge-invariant operators discussed here and the full operator algebra of the Γ model. We now proceed to the proof of this fact.[8] First, let us observe that relations (5.4) are exactly as in the definition of the Heisenberg group H_Q associated to a certain vector space

[8]Consult appendix A at this point.

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M of dimension $2n$, equipped with a quadratic form Q. Images of K and $\{\mathfrak{s}_g(e_i)\}_{i=2}^{|E|}$ in M span an isotropic subspace of dimension $|E|$, so $\mathrm{Arf}(Q) = 0$. Element $z \in H_Q$ acts as multiplication by -1, so H_Q is represented faithfully. Hence all relations satisfied in the algebra of gauge-invariant operators follow already from (5.4). This means that equations (5.5), (5.6) well-define an injective homomorphism of algebras. Dimensional considerations show that this homomorphism is also surjective and that $\mathcal{H}$ is isomorphic to a single copy of the standard representation of H_Q.

Since the algebra of gauge invariant operators is isomorphic to $\mathrm{End}(\mathcal{H})$, it is possible to construct an operator corresponding to $\Gamma(v,e)$ for any vertex v and any $e \in \mathrm{St}(v)$. It is the product of K and some number of $W(\epsilon_i)$ and $\mathfrak{s}_g(e_i)$, which is typically highly nonlocal.

5.2 Classification of Gauss' operators

In the subsection 5.1 we have considered a specific form of Gauss' operators motivated by our study of the Γ model. In this subsection we define and classify a larger class of gauge theories. This puts previous findings in a broader context and can be applied to discuss issues with locality of our models. We are interested in gauge theories with fermionic degrees of freedom on vertices and Ising degrees of freedom $U(e)$, $W(e)$ on edges. The full Hilbert space is assumed to be endowed with a unitary representation of the group of gauge transformations, i.e. for every vertex v there is given a unitary operator $G(v)$ such that $G(v)^2 = 1$ and $G(v)G(v') = G(v')G(v)$. Furthermore, we would like fermionic operators and $U(e)$ to transform under gauge transformations in the same way as in the conventional $\mathbb{Z}_2$ gauge theory, so that Wilson lines which are either closed or terminate at charges are gauge-invariant operators. This condition implies that $G(v)$ has to be of the form $\gamma(v)R(v)W(\delta v)$, where $R(v)$ is a function of operators $U(e)$ only. For simplicity we shall assume that $G(v)$ are of particularly simple form

$$G(v) = (-1)^{(\mu,v)}\gamma(v)U(\mathcal{T}v)W(\delta v), \tag{5.7}$$

with some $\mu \in C_0$ and $\mathcal{T} \in \mathrm{Hom}(C_0, C_1)$. Condition $G(v)^2 = 1$ implies that $\mathcal{T}$ has to satisfy $(\partial\mathcal{T}v, v) = 0$. Equation $G(v)G(v') = G(v')G(v)$ (for $v, v' \in V$) is equivalent to $(v, \partial\mathcal{T}v') = (v', \partial\mathcal{T}v)$. Thus $\partial\mathcal{T}$ is alternating, i.e. $(\theta, \partial\mathcal{T}\theta) = 0$ for every $\theta \in C_0$.

Theories with Gauss' operators related by a canonical transformation of the Heisenberg group generated by $\{U(e), W(e)\}_{e\in E}$ will be regarded as equivalent. This is a weak form of equivalence, since the allowed canonical transformations may be strongly non-local. Nevertheless it is true that equivalent theories have isomorphic algebras of gauge-invariant operators, since canonical transformations are implementable on representations of the Heisenberg group.

We wish to preserve the form of holonomy operators ($U(\tau)$ for $\partial\tau = 0$), so we consider canonical transformations of the form

$$U(e) \mapsto U(e), \qquad W(e) \mapsto (-1)^{(\theta,e)}U(\mathcal{S}e)W(e), \tag{5.8}$$

for $\theta \in C_0$ and $\mathcal{S} \in \mathrm{Hom}(C_1, C_1)$. In order for this to define a canonical transformation, $\mathcal{S}$ must be alternating. Under a transformation of this form, $\mathcal{T}$ changes according to

$$\mathcal{T} \mapsto \mathcal{T}' = \mathcal{T} + \mathcal{S}\delta, \tag{5.9}$$

while μ changes to some μ', which we ignore for now. It is easy to check that $\partial\mathcal{S}\delta$ is indeed automatically alternating if $\mathcal{S}$ is. Therefore the space of equivalence classes of allowed $\mathcal{T}$ is the quotient $\mathcal{Z}/\mathcal{B}$, where

$$\mathcal{Z} = \{\mathcal{T} : C_0 \to C_1 |\ \partial\mathcal{T} \text{ is alternating}\}, \tag{5.10a}$$

$$\mathcal{B} = \{\mathcal{T} : C_0 \to C_1 |\ \text{exists } \mathcal{S} : C_1 \to C_1 \text{ alternating and such that } \mathcal{T} = \mathcal{S}\delta\}. \tag{5.10b}$$

Now consider the class of Gauss' operators with a fixed $\mathcal{T}$. They are parameterized by chains $\mu \in C_0$. However there is a residual freedom of canonical transformations with $\mathcal{S} = 0$ and arbitrary θ. Under such transformations μ changes to $\mu' = \mu + \partial\theta$. Therefore there are two non-equivalent choices of μ, corresponding to two elements of $C_0/B_0 \cong \mathbb{Z}_2$.

We claim that the dimension of $\mathcal{Z}/\mathcal{B}$ is equal to $\dim(Z_1)$. For clarity we postpone the proof of this until the next paragraph. We will now establish a concrete one-to-one correspondence between pairs $(\tau, \alpha) \in Z_1 \times \mathbb{Z}_2$ and equivalence classes of Gauss' operators. For a given (τ, α) we choose a vertex $v_1 \in V$ and define:

$$G(v) = \begin{cases} \gamma(v)W(\delta v) & \text{for } v \neq v_1, \\ (-1)^\alpha \gamma(v)U(\tau)W(\delta v) & \text{for } v = v_1. \end{cases} \tag{5.11}$$

With this definition one has

$$\prod_{v \in V} G(v) = (-1)^\alpha \gamma \cdot U(\tau). \tag{5.12}$$

These elements are invariant with respect to canonical transformations of the form (5.8), which demonstrates that distinct pairs (τ, α) give Gauss' operators in different equivalence classes. Since the number of elements of $Z_1 \times \mathbb{Z}_2$ is equal to the number of equivalence classes, the one-to-one correspondence is established. There are two conclusions from this result that we would like to emphasize. Firstly, every equivalence class can be represented by $\mathcal{T}$ such that $\partial\mathcal{T}$ is not only alternating, but actually vanishes. Secondly, each equivalence class is uniquely characterized by the corresponding value of the "global" gauge transformation operator $\prod\limits_{v \in V} G(v)$, and thus by $\tau = \sum\limits_{v \in V} \mathcal{T}v$ and α. If $\partial\mathcal{T} = 0$, one has $\alpha = \sum\limits_{v \in V} (\mu, v)$.

In the remainder of this subsection we calculate the dimension of $\mathcal{Z}/\mathcal{B}$. First notice that $\mathcal{B}$ may be identified with the quotient of the space of alternating $\mathcal{S} : C_1 \to C_1$ by the subspace of those $\mathcal{S}$ for which $\mathcal{S}\delta = 0$. The former space has dimension $\frac{|E|(|E|-1)}{2}$. As for the latter, any of its elements satisfies also $\partial\mathcal{S} = 0$. Therefore it may be regarded as an alternating map $C^1/B^1 \to Z_1$. Since $C^1/B^1 \cong Z_1^*$, the pertinent dimension is equal to $\frac{\dim(Z_1)(\dim(Z_1)-1)}{2}$. Hence

$$\dim(\mathcal{B}) = \frac{|E|(|E|-1)}{2} - \frac{\dim(Z_1)(\dim(Z_1)-1)}{2}. \tag{5.13}$$

It remains to find the dimension of $\mathcal{Z}$. We consider the linear map

$$L_\partial : \mathrm{Hom}(C_0, C_1) \ni \mathcal{T} \longmapsto \partial\mathcal{T} \in \mathrm{Hom}(C_0, B_0). \tag{5.14}$$

Clearly L_∂ is surjective. Secondly, $\ker(L_\partial) = \mathrm{Hom}(C_0, Z_1)$, so $\dim\ker(L_\partial) = \dim(Z_1) \cdot |V|$.

Next consider the space $\mathfrak{R} = \{\mathcal{R} \in \mathrm{Hom}(C_0, B_0) | \ \mathcal{R} \text{ is alternating}\}$. Choosing V as a basis of C_0, elements of $\mathfrak{R}$ are represented as symmetric $|V| \times |V|$ matrices with zeros on the diagonal and such that sum of entries in every column is 0 (mod 2). Simple counting[9] shows that $\dim(\mathfrak{R}) = \frac{1}{2}(|V|-1)(|V|-2)$. Since $\mathcal{Z} = L_\partial^{-1}\mathfrak{R}$, we get

$$\dim \mathcal{Z} = \dim(\ker L_\partial) + \dim(\mathfrak{R}) = |V| \dim(Z_1) + \frac{1}{2}(|V|-1)(|V|-2). \tag{5.15}$$

Finally we use the fact that $\dim(Z_1) = |E| - |V| + 1$ to simplify

$$\dim(\mathcal{Z}/\mathcal{B}) = \dim(\mathcal{Z}) - \dim(\mathcal{B}) = \dim(Z_1). \tag{5.16}$$

5.3 Local formulations

Gauge theories defined by Gauss' operators of the form (5.11) are unsatisfactory for two reasons: firstly, one of the vertices is clearly distinguished in their formulation. Secondly, Gauss' operators are typically horribly non-local. Nevertheless, in many cases it is possible to remove this problem by a canonical transformation. We will now discuss how to do this in general and then specialize to the case $\tau = \zeta$.

Now suppose that τ is the boundary of a 2-chain ξ. Let F_ξ be the set of those $f \in F$ such that $(\xi, f) = 1$. For every $f \in F_\xi$ choose one vertex $v_f \in V$ incident to f. Define

$$\mathcal{T}v = \sum_{f \in F_\xi} \delta_{v,v_f} \cdot \partial f. \tag{5.17}$$

Then one has $\partial \mathcal{T} = 0$ and $\sum_{v \in V} \mathcal{T}v = \tau$. Furthermore, $\mathcal{T}v$ is at most the sum of some number of faces incident to the vertex v. Thus Gauss' operators are local and belong to the equivalence class specified by the cycle τ.

The above discussion raises the question whether the outlined construction can be carried out for $\tau = \zeta$, leading to a local $\mathbb{Z}_2$ gauge theory equivalent to the Γ model. Clearly this is always true for lattices representing simply-connected spaces, and more generally spaces X such that the homology group $H_1(X, \mathbb{Z}_2)$ is trivial. Otherwise one has to ask whether ζ represents a nontrivial homology class. Interestingly, it is known [42] that for a triangulation of a d-dimensional manifold X which is obtained by barycentric subdivision of another triangulation, simplicial cycle ζ is Poincaré dual to the $(d-1)$-st Stiefel-Whitney class of X. However, the restriction to a very specific class of cell decompositions is important here. In general it is not possible to determine the homology class of ζ in terms of the topology of X alone — it depends on the choice of decomposition. We will demonstrate this using the example of d-dimensional tori with arbitrary d. In this case all Stiefel-Whitney classes are trivial (since tori are parallelizable), but there exist decompositions for which ζ represents a nontrivial class, as well as such that ζ can be very explicitly trivialized. Indeed, for decompositions considered in subsection 4.4, cycle ζ is a boundary if and only if at least two L_i are even. If this condition is met, it is

[9] The number of free parameters in the first column is equal to $|V|-2$, since the first entry vanishes and the second one is determined in terms of the other by the requirement that the sum is even. In every subsequent column the number of free parameters decreases by one because the matrix is symmetric.

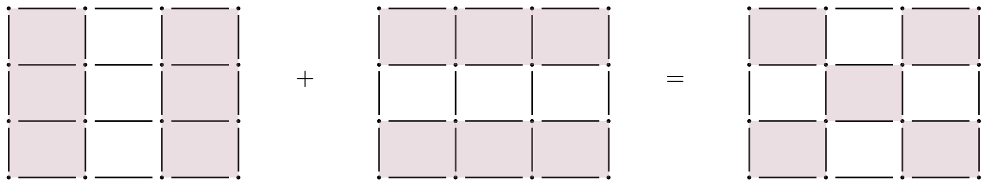

Figure 5. Trivialization of ζ for a two-dimensional torus: ζ is the boundary of the sum of shaded faces, which are arranged in a pattern resembling a chessboard. Up to switching the roles of white and grey squares this is the only possibility in the two-dimensional case.

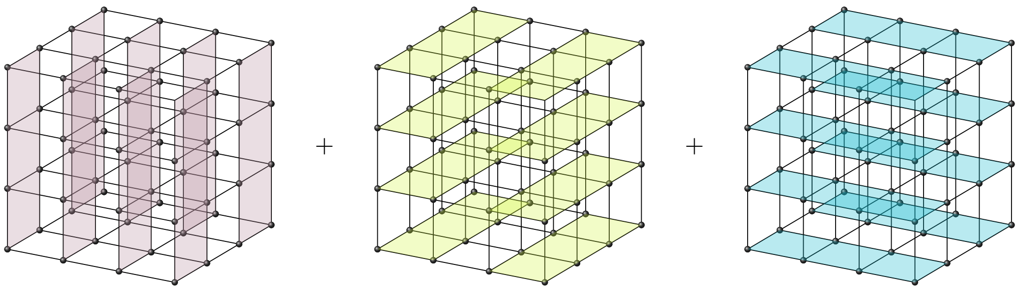

Figure 6. Particular trivialization of ζ for a three-dimensional torus: ζ is the boundary of the sum of all colored faces. Taking all faces of one color only obtains the sum of all edges in one direction.

possible to construct trivializations of ζ invariant with respect to all translations by an even number of lattice sites. This is illustrated in figures 5 and 6 for dimensions two and three, respectively. Analogous construction works in any dimension. Using the notation of subsection 4.4, Gauss' operators in the two-dimensional take the form

$$G(v) = (-1)^{(\eta,v)}\gamma(v) \cdot \begin{cases} U(\mathrm{NE}(v))W(\delta v) & \text{for v even,} \\ W(\delta v) & \text{for v odd,} \end{cases} \tag{5.18}$$

where η is any 0-chain with $\sum\limits_v (\eta, v) = \alpha$ and $\mathrm{NE}(v)$ is the plaquette to the north-east of v, i.e. the plaquette which has v as its south-west corner.

6 Duality with higher gauge theory

In this section we show how the Γ model may be dualized to a $(d-1)$-form $\mathbb{Z}_2$ gauge theory, again with a modified Gauss' law. More precisely, we will dualize only operators $\Gamma_*(v)$ and $S(e)$, which generate the commutant of constraint operators arising in bosonization. Composition of this map with the correspondence between the Γ model and fermions yields bosonization introduced in [13, 14, 34].

Hilbert space for the model we need is the tensor product of two-dimensional spaces associated to edges of the lattice. For each edge we introduce Pauli matrices $\{\sigma_i(e)\}_{i=1}^3$. We will think of $\sigma_3(e)$ as a higher dimensional parallel transport over a $(d-1)$-cell of the dual lattice.[10] As in ordinary gauge theory, parallel transports over individual cells will turn out not to be gauge-invariant. To construct an observable one has to take the product over all $(d-1)$-cells of some $(d-1)$-cycle. The simplest choice is the boundary of a d-cell, which corresponds to a vertex v of the original lattice. This gives the operator

$$\mathsf{h}(v) = \prod_{e \,:\, v \in e} \sigma_3(e). \tag{6.1}$$

Following [13, 14, 34], we would like to map operators $\Gamma_*(v)$ of the Γ model to $\mathsf{h}(v)$. This is possible only upon restricting to the subspace defined by the condition $\prod_{v \in V} \Gamma_*(v) = 1$, because the product of all $\mathsf{h}(v)$ is equal to 1 identically. In order to accommodate for existence of other states, we modify the mapping slightly by putting

$$\Gamma_*(v) \mapsto (-1)^{(\varepsilon,v)} \mathsf{h}(v) \tag{6.2}$$

for some 0-chain ε. We will think of ε as a d-cochain on the dual lattice, or an external d-form field. What truly matters here is the quantity $\beta = \sum_{v \in V} (\varepsilon, v)$, which characterizes the cohomology class of ε. Choices of ε with the same β give operator mapping related by conjugation with the product of some number of $\sigma_1(e)$ operators.

Next we construct an operator corresponding to $S(e)$. We consider the Ansatz

$$S(e) \mapsto \mathsf{e}(e) = \sigma_1(e) \cdot \prod_{e'} \sigma_3(e')^{\nu(e,e')} \tag{6.3}$$

for some function $\nu : E \times E \to \mathbb{Z}_2$. Since $S(e)^2 = -1$, we need to have $\mathsf{e}(e)^2 = -1$. This yields the condition $\nu(e,e) = 1$, which will be assumed from now on.

With the above definitions, braiding relations between $\mathsf{h}(v)$ and $\mathsf{e}(e)$ operators are correct, but we still need to impose braiding relations between distinct $\mathsf{e}(e)$. The sought-after condition is $\mathsf{e}(e)\mathsf{e}(e') = (-1)^{(\partial e, \partial e')} \mathsf{e}(e')\mathsf{e}(e)$, which translates to

$$\nu(e,e') + \nu(e',e) = (\partial e, \partial e'). \tag{6.4}$$

If edges e, e' do not share a common vertex, the above relation asserts that $\nu(e,e') = \nu(e',e)$. It seems to be most natural to put

$$\nu(e,e') = \nu(e',e) = 0 \qquad \text{if} \quad (\partial e, \partial e') = 0, \quad e \neq e'. \tag{6.5}$$

With this requirement, operators $\mathsf{e}(e)$ consist only of Pauli matrices on edges e' in the nearest vicinity of e, assuring that the mapping is local. On the other hand, if e and e' share one common vertex, values of $\nu(e,e')$ and $\nu(e',e)$ have to be opposite. In other words, we have to choose either $\nu(e,e') = 1$ and $\nu(e',e) = 0$, or vice versa.

In the above we have argued that functions ν satisfying (6.4) exist and may be subjected to the additional locality condition (6.5). Next, we will demonstrate that it is essentially

[10]Strictly speaking it is not necessary to invoke the dual lattice to formulate this model. Indeed, all formulas that follow will be written in terms of cells of the original lattice. However, their gauge theoretic interpretation is most directly seen on the dual lattice.

unique, in the sense that distinct choices yield operator maps related by a local unitary rotation. Indeed, given two such functions ν_1, ν_2 we put $\omega(e,e') = \nu_1(e,e') + \nu_2(e,e')$. Then $\omega(e,e) = 0$ and $\omega(e,e') = \omega(e',e)$. Transformation

$$\sigma_1(e) \mapsto \sigma_1(e) \cdot \prod_{e'} \sigma_3(e')^{\omega(e,e')}, \qquad \sigma_3(e) \mapsto \sigma_3(e) \tag{6.6}$$

defines an algebra automorphism, see appendix A. It is local if both ν_1 and ν_2 satisfy (6.5). By construction, it is such that

$$\mathsf{h}(v) \mapsto \mathsf{h}(v), \qquad \sigma_1(e) \cdot \prod_{e'} \sigma_3(e')^{\nu_1(e,e')} \mapsto \sigma_1(e) \cdot \prod_{e'} \sigma_3(e')^{\nu_2(e,e')}. \tag{6.7}$$

In a similar way, any signs that could be included in the definition of $\mathsf{e}(e)$ could also be reabsorbed by a unitary transformation, so we do not consider including them. Besides the braiding relations, there exist certain global constraints that have to be taken into account. We have already mentioned that the operator $\Gamma_* = \prod\limits_{v \in V} \Gamma_*(v)$ is sent to the c-number $(-1)^\beta$, so this mapping may be valid only upon restricting to the corresponding subspace of the Γ model. On the other hand, by (4.26), we have that on this subspace $S(\ell) = (-1)^{\alpha+\beta}$ for an Eulerian circuit $\ell = (e_1, \dots, e_n)$. This gives a constraint

$$\mathsf{e}(e_1) \dots \mathsf{e}(e_n) = (-1)^{\alpha+\beta}, \tag{6.8}$$

which is a nontrivial restriction, since the left hand side is not a c-number. It is not difficult to check that this equation is satisfied on a subspace of dimension $2^{|E|-1}$, which is also equal to the dimension of the Hilbert space of the Γ model with fixed value of Γ_*.

Using methods and results of previous sections, it is easy to check that there are no further independent relations satisfied by operators $\Gamma_*(v)$ and $S(e)$. Therefore, the proposed map well-defines a homomorphism of operator algebras.

Construction of the duality is now essentially completed. We will now give the correspondence between various notions formulated in the two pictures of the model.

Firstly, for every circuit $\ell = (e_1, \dots, e_n)$ the operator $S(\ell)$ is mapped to a certain operator $\mathsf{e}(\ell)$. Its explicit form is easy to evaluate:

$$\mathsf{e}(\ell) = (-1)^{\sum\limits_{i<j} \nu(e_i,e_j)} \cdot \prod_{i=1}^{n} \sigma_1(e_i) \cdot \prod_{e'} \sigma_3(e')^{\sum\limits_{i=1}^{n} \nu(e_i,e')}. \tag{6.9}$$

This operator commutes with all $\mathsf{h}(v)$ and $\mathsf{e}(e)$ and satisfies $\mathsf{e}(\ell)^2 = 1$. In the case that ℓ is the loop around the boundary of a face f, we interpret $\mathsf{e}(\ell)$ as a Gauss' operator of the gauge theory and denote it by $\mathsf{g}(f)$. Indeed, it satisfies the expected relation

$$\mathsf{g}(f)\sigma_3(e) = (-1)^{(\partial f, e)} \sigma_3(e) \mathsf{g}(f). \tag{6.10}$$

States violating the constraint $S(\ell) = 1$ were interpreted earlier in terms of fermions propagating in an external $\mathbb{Z}_2$ gauge field A. On the higher gauge theory side this background field is thought of as a $(d-2)$-form electric charge distribution, localized on the cycle Poincaré dual to δA. Indeed, for these states we have $\mathsf{g}(f) = (-1)^{(\delta A, f)}$.

Operators $\mathsf{e}(\ell)$ with ℓ non-contractible furnish a $(d-1)$-form $\mathbb{Z}_2$ symmetry of the higher gauge theory. They act trivially on all $\mathsf{h}(v)$, but not so on holonomies along homologically nontrivial $(d-1)$-cycles of the dual lattice. Flat background gauge fields for fermions correspond to eigenspaces of this symmetry. The symmetry is subject to a 't Hooft anomaly, whose form was identified in [43]. One may understand the presence of the anomaly in an intuitive way as follows. We see from (6.9) that the symmetry is not on-site (the correct notion of a site being an edge, i.e. a dual $(d-1)$-cell). Thus there is no canonical gauging procedure, but one can attempt to take a different route. Symmetry operators $\mathsf{e}(\ell)$ furnish a representation of the group of 1-cycles (dual $(d-1)$-cocycles). We would like to extend it to an action of the group all 1-chains (dual $(d-1)$-cochains). The simplest choice would be to take the operator corresponding to the transformation at the edge e to be $\mathsf{e}(e)$, but this does not work, since $\mathsf{e}(e)$ placed at different edges commute only up to signs. It is expected that there is no way around this difficulty.

We remark that in [13, 14, 34] a more specific construction of duality between fermions and higher gauge theory, essentially corresponding to a particular choice of the function ν, was presented. It is formulated in terms of higher cup products [44] and depends on a choice of a branching structure[11] on the dual lattice, assumed to be a triangulation. An interesting feature of this approach is that certain sign factors appearing in constraint operators may be expressed in terms of a cocycle w_2 representing the second Stiefel-Whitney class. Given a spin structure, understood as a cochain E such that[12] $\delta E = w_2$, one may absorb these signs by redefining fermionic kinetic operators. The fact that signs may be shuffled between the definition of the bosonization map and the form of Gauss' operator may be traced to the duality between background gauge fields and background electric charge distributions. We remark that the role of spin structures in bosonization has been discussed also in [45].

Despite the elegance of the construction outlined above, we would like to emphasize that existence of a spin structure is not necessary to construct bosonization maps. In fact all models considered in this paper make sense on a large class of graphs, which are not necessarily discretizations of manifolds and hence do not have well-defined Stiefel-Whitney classes or spin structures. Of course, it may very well be true that spin structures do play an indispensable role if one imposes some *naturality* or *functoriality* conditions on duality maps, but it is not completely clear to us what would be the correct formulation of this. On the other hand, spin structures clearly become important in concrete dynamical models. For example, continuum limits of many lattice models with fermions should depend on a spin structure. Another interesting example of this is the discussion of the Gu-Wen model [46] in [43].

7 Summary and outlook

We generalized the bosonization prescription based on the Γ model, presented a new proof of its correctness and reinterpreted it in terms of lattice $\mathbb{Z}_2$ gauge theory. We found that

[11] Branching structure is a choice of orientations of edges such that there is no loop in any triangle.

[12] In the notation of papers we are refering to, this equation takes the form $\delta E = w_2$, but in the two perspectives the roles of the lattice and its dual are reversed, so operators δ and ∂ are exchanged.

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its alternative gauge-theoretic description involves modified Gauss' operators, much as in Chern-Simons-like theories. Furthermore, we discussed the duality with higher gauge theories recently proposed in the context of bosonization. These results are valid independently of the spatial dimension.

In order to actually perform bosonization (rather than to couple fermions to gauge fields) it is necessary to introduce constraints in the Γ model. They can be interpreted as a flatness condition for the gauge field. We have presented a solution of these constraints in the case of two-dimensional tori. Unfortunately, our method does not seem to generalize to higher dimensions in a straightforward way. Thus dealing with constraints in an efficient way for general geometries remains a challenge for future research.

Another interesting problem not solved for now is to obtain a useful state-sum formulation of the Γ model. Furthermore, it remains to be seen whether it is possible to apply constructions of this type to shed new light on some problems in lattice gauge theory, such as those related to fermion doubling or anomalous symmetries.

A Canonical transformations for Ising degrees of freedom

In this appendix we summarize properties of the Heisenberg groups for $\mathbb{Z}_2$-valued degrees of freedom and their automorphisms, called canonical transformations. There are essentially no new results here, but we do not know a reference in which the whole material presented here is discussed concisely. We refer to [47] and [48] for further discussion.

Let M be a finite-dimensional $\mathbb{Z}_2$-vector space. A bilinear form $\Omega : M \times M \to \mathbb{Z}_2$ is said to be alternating if $\Omega(x, x) = 0$ for every $x \in M$. Every alternating form is symmetric, but the converse is not true.[13] Alternating form which is non-degenerate, i.e. such that for every $x \in M$ there exists $y \in M$ such that $\Omega(x, y) = 1$, is called a symplectic form. If Ω is a symplectic form, there exists a basis $\{e_i, f_i\}_{i=1}^n$ in which Ω takes the canonical form

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \qquad \Omega(e_i, f_j) = \delta_{i,j}. \tag{A.1}$$

In particular, the dimension of M is necessarily even. It is the only invariant of (M, Ω).

Function $Q : M \to \mathbb{Z}_2$ is called a quadratic form if the map $\Omega : M \times M \to \mathbb{Z}_2$ given by $\Omega(x, y) = Q(x + y) - Q(x) - Q(y)$ is a bilinear form. Bilinear forms Ω arising this way are automatically alternating. If Ω is also non-degenerate, we say that Q is non-singular. We assume this condition from now on. Thus $\dim(M) = 2n$. Subspace $N \subseteq M$ is said to be isotropic if $Q(x) = 0$ for every $x \in N$. One can show that maximal isotropic subspaces of M are of dimension n or $n - 1$. These two possibilities correspond to values 0 and 1, respectively, of the so called Arf invariant $\mathrm{Arf}(Q)$ of Q [49]. Dimension of M and the Arf invariant are the only invariants of (M, Q). In the case $\mathrm{Arf}(Q) = 0$ it is possible to choose a basis in which Ω takes the canonical form (A.1) and additionally $Q(e_i) = Q(f_i) = 0$.

[13] It is true over any field that alternating forms are skew-symmetric, but in the case of fields of characteristic two skew-symmetry and symmetry is the same. Furthermore, it is true in general that skew-symmetry of a form Ω implies that $2\Omega(x, x) = 0$ for every $x \in M$. This implies that Ω is alternating if 2 is invertible, but it is a vacuous statement in the case of characteristic two.

Let (M, Q) be as in the previous paragraph and let $\mathfrak{B} = \{x_i\}_{i=1}^{2n}$ be a basis of M. The Heisenberg group $H_{Q,\mathfrak{B}}$ is the group with generators $\{z\} \cup \{T_i\}_{i=1}^{2n}$, subject to relations

$$z^2 = 1, \qquad T_i^2 = z^{Q(x_i)}, \qquad zT_i = T_i z, \qquad T_i T_j = z^{\Omega(x_i,x_j)} T_j T_i. \tag{A.2}$$

Its center $Z(H_{Q,\mathfrak{B}})$ is generated by the element z. Quotient $H_{Q,\mathfrak{B}}/Z(H_{Q,\mathfrak{B}})$ is a $\mathbb{Z}_2$-vector space. It may be identified with M, with the coset of T_i corresponding to the element x_i. We let π be the canonical projection $H_{Q,\mathfrak{B}} \to M$. It is easy to check that $gg' = z^{\Omega(\pi(g),\pi(g'))} g' g$ and $g^2 = z^{Q(\pi(g))}$ for every $g, g' \in H_{q,\mathfrak{B}}$.

Suppose M' is another $\mathbb{Z}_2$-vector space and let $\varphi : (M, Q) \to (M', Q')$ be an isometry, i.e. a linear map such that $Q'(\varphi(x)) = Q(x)$ for every $x \in M$. Choose a basis $\mathfrak{B}'$ of M' and consider the group $H_{Q',\mathfrak{B}'}$. For each x_i we can find a (non-unique) $T'_i \in H_{Q',\mathfrak{B}'}$ such that $\pi(T'_i) = \varphi(x'_i)$. Elements T'_i satisfy all relations obeyed by T_i, so there is a unique group homomorphism $\Phi : H_{Q,\mathfrak{B}} \to H_{Q',\mathfrak{B}'}$ such that

$$\Phi(z) = z, \qquad \Phi(T_i) = T'_i. \tag{A.3}$$

Clearly Φ is a lift of φ, in the sense that $\pi \circ \Phi = \varphi \circ \pi$. We emphasize that homomorphisms Φ lifting φ are not unique, because in the above constructions we have to choose elements T'_i, with distinct choices corresponding to distinct lifts. This means that Heisenberg groups corresponding to (M, Q) constructed using different bases of M are isomorphic, but not canonically isomorphic.[14] Having said that, we will abuse the notation slightly by abbreviating $H_{Q,\mathfrak{B}}$ to H_Q.

In this paper we will use only quadratic forms Q with $\mathrm{Arf}(Q) = 0$. In this case we can choose a basis of M in which Q takes the canonical form. We let $U_i, W_i \in H_Q$ be some lifts of e_i and f_i, respectively.

Automorphisms of H_Q will be called canonical transformations. Every canonical transformation Φ acts trivially on $Z(H_Q)$, hence induces an isometry φ of (M, Q). The map $\widetilde{\pi} : \Phi \mapsto \varphi$ is a homomorphism from $\mathrm{Aut}(H_Q)$ to $\mathrm{O}(M, Q)$, the orthogonal group of Q. It is clear from the preceding discussion that $\widetilde{\pi}$ is surjective. Next, let Φ be in the kernel of $\widetilde{\pi}$. Then we have $\pi \circ \Phi(U_i) = e_i$ and $\pi \circ \Phi(W_i) = f_i$, so

$$\Phi(U_i) = (-1)^{a_i} U_i, \qquad \Phi(W_i) = (-1)^{b_i} W_i \tag{A.4}$$

for some $a_i, b_i \in \mathbb{Z}_2$. Conversely, for every collection $\{a_i, b_i\}_{i=1}^{n}$ the above formula defines a canonical transformation $\Phi \in \ker(\widetilde{\pi})$. Automorphisms of this form are precisely the inner automorphisms: $\Phi(g') = gg'g^{-1}$ with $g = \prod_{i=1}^{n} U_i^{a_i} W_i^{b_i} \in H_q$. Therefore $\ker(\widetilde{\pi})$ may be identified with M, since $Z(H_Q)$ is precisely the group of those elements of H_Q which act trivially on H_Q. We have shown that $\mathrm{Aut}(H_Q)$ is an extension of $\mathrm{O}(M, Q)$ by M. Interestingly, it is known that this extension is non-split for $n \geq 3$ [50]. This is in contrast with the more standard situation for Heisenberg groups in characteristic different than two.

The last issue we need to touch upon is representation theory. First we define the standard representation ρ of H_Q on $(\mathbb{C}^2)^{\otimes n}$ by

$$\rho(z) = -1, \quad \rho(U_i) = 1_{\mathbb{C}^2}^{\otimes(i-1)} \otimes \sigma_3 \otimes 1_{\mathbb{C}^2}^{\otimes(n-i)}, \quad \rho(W_i) = 1_{\mathbb{C}^2}^{\otimes(i-1)} \otimes \sigma_1 \otimes 1_{\mathbb{C}^2}^{\otimes(n-i)}, \tag{A.5}$$

[14]There is a canonically defined class of isomorphisms modulo compositions with inner automorphisms.

where $\{\sigma_i\}_{i=1}^3$ are the Pauli matrices. It is easy to see that this representation is irreducible. We claim that up to isomorphism this is the only irreducible representation of H_Q on which $Z(H_Q)$ acts nontrivially. Indeed, representations on which $Z(H_Q)$ acts trivially are in one-to-one correspondence with representations of M. Now recall [51] that the number of non-isomorphic irreducible representations of a finite group is equal to the number of its conjugacy classes. It is easy to check that the number of conjugacy classes in H_Q exceeds the number of conjugacy classes in M by one, which completes the argument. The statement just proven is an analogue of the Stone-von Neumann theorem for Ising degrees of freedom. It has an additional corollary that every non-trivial normal subgroup of H_Q contains z.

Now let Φ be a canonical transformation. Then $\rho \circ \Phi$ is also an irreducible representation on which the center acts nontrivially, so by the above theorem there exists a unitary endomorphism $p(\Phi)$ of the standard module, unique up to phase, such that $\rho(\Phi(g)) = p(\Phi)\rho(g)p(\Phi)^{-1}$ for every $g \in H_Q$. Assignment $\Phi \mapsto p(\Phi)$ is a projective representation of the group of canonical transformations. Even its restriction to $M \subseteq \mathrm{Aut}(H_Q)$ is not equivalent to a linear representation. It can be lifted to a genuine representation of a certain finite central extension of $\mathrm{Aut}(H_Q)$. The structure of this extension is not known to us, but fortunately it will not be needed. The important point is the existence of p.

B Graphs with vertices of odd degree

In this appendix we shall briefly describe a generalization of the Γ model to the case in which some vertices have odd degree. It will be shown that this has the effect of introducing additional degrees of freedom on each vertex of odd degree. We construct operators which create and annihilate these excitations.

We decompose the set of vertices V into two disjoint sets V_α of vertices of degree α (mod 2). Hilbert spaces associated to vertices of even degree are constructed as earlier. For vertices v of odd degree the Clifford algebra generated by $\{\Gamma_*(v)\} \cup \{\Gamma(v,e)\}_{e\in \mathrm{St}(v)}$ has one (rather than two) non-isomorphic irreducible representation. We take this representation as the Hilbert space associated to v. In contrast to the even case, $\Gamma_*(v)$ cannot be expressed in terms of other generators. With this definition, the dimension of the full Hilbert space $\mathcal{H}$ is

$$\dim(\mathcal{H}) = \prod_{v\in V_0} 2^{\frac{\deg(v)}{2}} \prod_{v\in V_1} 2^{\frac{\deg(v)+1}{2}} = 2^{|E|+\frac{1}{2}|V_1|}. \tag{B.1}$$

Since this is an integer, it follows that $|V_1|$ is even. This can also be seen by reducing the equation $\sum_{v\in V} \deg(v) = 2|E|$ modulo two.

As in the case of graphs with even vertices only, we can decompose $\mathcal{H} = \bigoplus_{[A]\in Z_1^*} \mathcal{H}_{[A]}$. Calculation analogous to the one in equation (4.7) shows that each $\mathcal{H}_{[A]}$ has the same dimension. Since the number of distinct $[A]$ is $2^{|E|-|V|+1}$, one has

$$\dim(\mathcal{H}_{[A]}) = \frac{2^{|E|+\frac{1}{2}|V_1|}}{2^{|E|-|V|+1}} = 2^{|V|-1} \cdot 2^{\frac{1}{2}|V_1|}. \tag{B.2}$$

Thus $\mathcal{H}_{[A]}$ is as large as $2^{\frac{1}{2}|V_1|}$ "halves" of the Fock space. To account for this multiplicity we introduce new operators. Let $\ell = (e_1, \ldots, e_n)$ be a path with initial point $v = s(e_1)$ and final point $v' = t(e_n)$ of odd degrees. We define

$$\Psi(\ell) = i^{\frac{\deg(v)+\deg(v')}{2}+1} \left(\Gamma_*(v) \cdot \prod_{e \in \mathrm{St}(v)} \Gamma(v,e) \right) S(\ell) \left(\Gamma_*(v') \cdot \prod_{e \in \mathrm{St}(v')} \Gamma(v',e) \right), \tag{B.3}$$

where we choose some orderings of $\mathrm{St}(v)$ and $\mathrm{St}(v')$, modulo even permutations.

We list in points the main properties of $\Psi(\ell)$:

- $\Psi(\ell)$ commutes with $S(\ell')$ and $\Gamma_*(v'')$ for any path ℓ' and any vertex v''.
- If ℓ' is a path with initial vertex v' and final vertex v'', then

  $$\Psi(\ell)\Psi(\ell') = \begin{cases} \Psi(\ell\ell') & \text{if } v \neq v'', \\ S(\ell\ell') & \text{if } v = v'', \end{cases} \tag{B.4}$$

  where $\ell\ell'$ is the concatenation of ℓ and ℓ'.
- We have braiding relations

  $$\Psi(\ell)\Psi(\ell') = (-1)^{(\partial[\ell],\partial[\ell'])}\Psi(\ell')\Psi(\ell). \tag{B.5}$$
- $\Psi(\ell)^2 = -1$.

One can further decompose each $\mathcal{H}_{[A]}$ into subspaces corresponding to even and odd numbers of fermions, $\mathcal{H}_{[A],0}$ and $\mathcal{H}_{[A],1}$. One can show that each $\mathcal{H}_{[A],\alpha}$ is an irreducible representation (of dimension $2^{|V|+\frac{1}{2}|V_1|-2}$) of the algebra $\mathcal{A}_0 \otimes_{\mathbb{C}} \mathbb{C}\mathcal{G}$, where $\mathbb{C}\mathcal{G}$ is the group algebra of the group $\mathcal{G}$ generated by all Ψ operators.

There are some similarities between the presented structure and the so-called delocalized fermions [11, 52], considered e.g. in the field of topological quantum computation. These excitations consist of multiple fermionic degrees of freedom, located at different lattice sites and connected by strings.

Acknowledgments

Initial stage of this project has been realized with J. Wosiek and A. Wyrzykowski. We thank J. Wosiek for an introduction to the subject of bosonization, discussions and encouragement. We are grateful to Y-A. Chen, A. Francuz, L. Hadasz, Z. Komargodski, M. Rocek, K. Roumpedakis and S. Seifnashri for discussions. Analysis carried out in section 6 has been suggested to us by an anonymous referee. BR was supported by the NCN grant UMO-2016/21/B/ST2/01492 and the MNS donation for PhD students and young scientists N17/MNS/000040.

References

[1] P. Jordan and E.P. Wigner, *Über das Paulische Äquivalenzverbot*, *Z. Phys.* **47** (1928) 631 [INSPIRE].

[2] E. Witten, *Non-abelian bosonization in two dimensions*, *Commun. Math. Phys.* **92** (1984) 455 [INSPIRE].

[3] D. Sénéchal, *An Introduction to Bosonization*, in *Theoretical Methods for Strongly Correlated Electrons*, D. Sénéchal, A.M. Tremblay and C. Bourbonnais eds., CRM Series in Mathematical Physics, Springer (2004).

[4] T.D. Schultz, D.C. Mattis and E.H. Lieb, *Two-Dimensional Ising Model as a Soluble Problem of Many Fermions*, *Rev. Mod. Phys.* **36** (1964) 856 [INSPIRE].

[5] S. Mandal and N. Surendran, *Exactly solvable Kitaev model in three dimensions*, *Phys. Rev. B* **79** (2009) 024426.

[6] A.O. Gogolin, A.A. Nersesyan and A.M. Tsvelik, *Bosonization and Strongly Correlated Systems*, Cambridge University Press (1998).

[7] A. Kapustin and R. Thorngren, *Fermionic SPT phases in higher dimensions and bosonization*, *JHEP* **10** (2017) 080 [arXiv:1701.08264] [INSPIRE].

[8] J. Condella and C.E. Detar, *Potts flux tube model at nonzero chemical potential*, *Phys. Rev. D* **61** (2000) 074023 [hep-lat/9910028] [INSPIRE].

[9] Y. Delgado, C. Gattringer and A. Schmidt, *Solving the sign problem of two flavor scalar electrodynamics at finite chemical potential*, *PoS* **LATTICE2013** (2014) 147 [arXiv:1311.1966] [INSPIRE].

[10] C. Gattringer, T. Kloiber and V. Sazanov, *Solving the sign problems of the massless lattice Schwinger model with a dual formulation*, *Nucl. Phys. B* **879** (2015) 732.

[11] A.Yu. Kitaev, *Fault-tolerant quantum computation by anyons*, *Annals Phys.* **303** (2003) 2 [quant-ph/9707021] [INSPIRE].

[12] A. Kitaev and C. Laumann, *Topological phases and quantum computation*, arXiv:0904.2771.

[13] Y.-A. Chen, A. Kapustin and Đ. Radičević, *Exact bosonization in two spatial dimensions and a new class of lattice gauge theories*, *Annals Phys.* **393** (2018) 234 [arXiv:1711.00515] [INSPIRE].

[14] Y.-A. Chen and A. Kapustin, *Bosonization in three spatial dimensions and a 2-form gauge theory*, *Phys. Rev. B* **100** (2019) 245127 [arXiv:1807.07081] [INSPIRE].

[15] J. Wosiek, *A local representation for fermions on a lattice*, *Acta Phys. Polon. B* **13** (1982) 543 [INSPIRE].

[16] C.P. Burgess, C.A. Lütken and F. Quevedo, *Bosonization in higher dimensions*, *Phys. Lett. B* **336** (1994) 18 [hep-th/9407078] [INSPIRE].

[17] P. Kopietz, *Bosonization of Interacting Fermions in Arbitrary Dimensions*, Springer (1997).

[18] S.B. Bravyi and A.Yu. Kitaev, *Fermionic Quantum Computation*, *Annals Phys.* **298** (2002) 210.

[19] R.C. Ball, *Fermions without Fermion Fields*, *Phys. Rev. Lett.* **95** (2005) 176407 [cond-mat/0409485] [INSPIRE].

JHEP12(2020)118

[20] F. Verstraete and J.I. Cirac, *Mapping local Hamiltonians of fermions to local Hamiltonians of spins*, J. Stat. Mech. **2005** (2005) P09012.

[21] E. Fradkin, *Jordan-Wigner transformation for quantum-spin systems in two dimensions and fractional statistics*, Phys. Rev. B **63** (1989) 322.

[22] A. Karch and D. Tong, *Particle-Vortex Duality from 3D Bosonization*, Phys. Rev. X **6** (2016) 031043 [arXiv:1606.01893] [INSPIRE].

[23] E. Zohar and J.I. Cirac, *Eliminating fermionic matter fields in lattice gauge theories*, Phys. Rev. B **98** (2018) 075119 [arXiv:1805.05347] [INSPIRE].

[24] A. Karch, D. Tong and C. Turner, *A web of 2d dualities: $\mathbb{Z}_2$ gauge fields and Arf invariants*, SciPost Phys. **7** (2019) 007 [arXiv:1902.05550] [INSPIRE].

[25] R. Thorngren, *Anomalies and Bosonization*, Commun. Math. Phys. **378** (2020) 1775 [arXiv:1810.04414] [INSPIRE].

[26] T. Senthil, D.T. Son, C. Wang and C. Xu, *Duality between $(2+1)d$ quantum critical points*, Phys. Rept. **827** (2019) 1 [arXiv:1810.05174] [INSPIRE].

[27] N. Seiberg, T. Senthil, C. Wang and E. Witten, *A duality web in $2+1$ dimensions and condensed matter physics*, Annals Phys. **374** (2016) 395 [arXiv:1606.01989] [INSPIRE].

[28] A.M. Szczerba, *Spins and fermions on arbitrary lattices*, Commun. Math. Phys. **98** (1985) 513 [INSPIRE].

[29] A. Bochniak, B. Ruba, J. Wosiek and A. Wyrzykowski, *Constraints of kinematic bosonization in two and higher dimensions*, Phys. Rev. D **102** (2020) 114502 [arXiv:2004.00988] [INSPIRE].

[30] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Commun. Math. Phys. **129** (1990) 393 [INSPIRE].

[31] D.S. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*, Commun. Math. Phys. **156** (1993) 435 [hep-th/9111004] [INSPIRE].

[32] Y. Wan, J.C. Wang and H. He, *Twisted gauge theory model of topological phases in three dimensions*, Phys. Rev. B **92** (2015) 045101 [arXiv:1409.3216] [INSPIRE].

[33] F. Wilczek, *Magnetic Flux, Angular Momentum, and Statistics*, Phys. Rev. Lett. **48** (1982) 1144 [INSPIRE].

[34] Y.-A. Chen, *Exact bosonization in arbitrary dimensions*, Phys. Rev. Res. **2** (2020) 033527 [arXiv:1911.00017] [INSPIRE].

[35] H.A. Kramers and G.H. Wannier, *Statistics of the Two-Dimensional Ferromagnet. Part I*, Phys. Rev. **60** (1941) 252 [INSPIRE].

[36] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002).

[37] J.L. Gross and J. Yellen, *Handbook of Graph Theory*, CRC Press (2003).

[38] J.A. Beachy, *Introductory Lectures on Rings and Modules*, Cambridge University Press (1999).

[39] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized global symmetries*, JHEP **02** (2015) 172 [arXiv:1412.5148] [INSPIRE].

[40] F.J. Wegner, *Duality in Generalized Ising Models and Phase Transitions Without Local Order Parameters*, J. Math. Phys. **12** (1971) 2259 [INSPIRE].

JHEP12(2020)118

[41] J.B. Kogut, *An introduction to lattice gauge theory and spin systems*, *Rev. Mod. Phys.* **51** (1979) 659 [INSPIRE].

[42] S. Halperin and D. Toledo, *Stiefel-Whitney homology classes*, *Annals Math.* **96** (1972) 511.

[43] D. Gaiotto and A. Kapustin, *Spin TQFTs and fermionic phases of matter*, *Int. J. Mod. Phys. A* **31** (2016) 1645044 [arXiv:1505.05856] [INSPIRE].

[44] N.E. Steenrod, *Products of Cocycles and Extensions of Mappings*, *Annals Math.* **48** (1947) 290.

[45] Đ. Radičević, *Spin Structures and Exact Dualities in Low Dimensions*, arXiv:1809.07757 [INSPIRE].

[46] Z.-C. Gu and X.-G. Wen, *Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ models and a special group supercohomology theory*, *Phys. Rev. B* **90** (2014) 115141 [arXiv:1201.2648] [INSPIRE].

[47] A. Weil, *Sur certains groupes d'opérateurs unitaires*, *Acta Math.* **111** (1964) 143.

[48] L. Blasco, *Paires duales réductives en caractéristique 2*, *Mém. Soc. Math. Fr.* **52** (1993) 1.

[49] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 270, Springer-Verlag, Berlin (1985) [DOI].

[50] R.L. Griess Jr., *Automorphisms of extraspecial groups and nonvanishing degree 2 cohomology*, *Pac. J. Math.* **48** (1973) 403.

[51] A.A. Kirillov, *Elements of the theory of representations*, Springer-Verlag (1976).

[52] S.M. Bhattacharjee, M. Mj and A. Bandyopadhyay eds., *Topology and Condensed Matter Physics*, Springer Singapore (2017).

Constraints of kinematic bosonization in two and higher dimensions

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(Received 2 October 2020; accepted 3 November 2020; published 4 December 2020)

Despite being less known, local bosonizations of fermionic systems exist in spatial dimensions higher than 1. Interestingly, the dual bosonic systems are subject to local constraints, as in theories with gauge freedom. These constraints effectively implement long distance exchange interactions. In this work, we study in detail one such system, proposed a long time ago. Properties of the constraints are elaborated for two-dimensional, rectangular lattices of arbitrary sizes. For several small systems, the constraints are solved analytically. It is checked that spectra of reduced spin Hamiltonians agree with the original fermionic ones. The equivalence is extended to fermions in the presence of background Wegner $\mathbb{Z}_2$ fields coupling to fermionic parity. This is illustrated by an explicit calculation for a particular configuration of Wegner's variables. Finally, a possible connection with the recently proposed web of dualities is discussed.

DOI: 10.1103/PhysRevD.102.114502

I. INTRODUCTION

Relation between fermionic and spin degrees of freedom is an old subject [1,2], but it still attracts a fair amount of interest. There is a variety of motivations for such studies. The presence of Grassmann variables in fermionic field theories leads to practical difficulties in their study, hence the desire to eliminate them [3,4]. Second, equivalences between apparently different physical systems often offer new insights into their dynamics. There has been a lot of progress in these directions recently. For instance, it has been shown [5,6] that fermions in space dimension d can be exactly mapped to a local generalized gauge theory on the dual lattice, with $\mathbb{Z}_2$ gauge variables associated to $(d-1)$-dimensional objects (hence an Ising model for $d=1$, standard gauge theory with modified Gauss' law for $d=2$ and the so-called higher gauge theories for $d\geq 3$). This idea has been motivated by studies of fermions in topological quantum field theories [7]. There exists also a variety of known dualities in the continuum, especially in low dimensions [8–10]. Many of them have been discovered in string theoretic considerations. Some of them connect bosons to fermions, which provides another point of view on bosonizaton. Finally, intensive studies of quantum computers and "quantum algorithms" stimulate some progress in the Hamiltonian formulation; see in particular [11–14].

Spin-fermion maps are particularly well understood and exploited in systems of spatial dimension one. Their extensions to higher dimensions typically lead to complicated nonlocal interactions or constraints and seems to be not practical.

In this paper, we revisit an old proposal [4,15] in which spins interact locally and satisfy local constraints. These constraints effectively take care of the nonlocality of fermions in arbitrary space dimensions.

Let us begin with a simple fermionic Hamiltonian on a one-dimensional lattice

$$H_f = i\sum_n (\phi(n)^\dagger\phi(n+1) - \phi(n+1)^\dagger\phi(n)), \quad (1)$$

with $\{\phi(m)^\dagger, \phi(n)\} = \delta_{mn}$. Its equivalent in terms of spin variables reads

$$H_s = \frac{1}{2}\sum_n (\sigma^1(n)\sigma^2(n+1) - \sigma^2(n)\sigma^1(n+1)), \quad (2)$$

where Pauli matrices $\sigma^k(n)$ commute between different sites labeled by n. Boundary conditions for σ^1 and σ^2 are taken to be opposite to (resp. the same as) boundary conditions for fermions if the number of fermions $\sum_n \phi(n)^\dagger\phi(n)$ is even (resp. odd). The standard way to derive this equivalence is via the Jordan-Wigner transformation [1]. Direct generalization of this method to higher dimensions leads to nonlocal spin-spin interactions. Therefore, we adopt another route, which applies also to multidimensional systems.

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To this end, we introduce the following Clifford variables (also called Majorana fermions):

$$X(n) = \phi(n)^\dagger + \phi(n), \qquad Y(n) = i(\phi(n)^\dagger - \phi(n)), \quad (3)$$

and rewrite the fermionic Hamiltonian from Eq. (1) in terms of link (or hopping) operators,

$$H_s = \frac{1}{2}\sum_n (S(n) + \tilde{S}(n)),$$
$$S(n) = iX(n)X(n+1), \qquad \tilde{S}(n) = iY(n)Y(n+1). \quad (4)$$

Link operators obey the following relations:

$$\begin{aligned} S(n)^2 &= 1, \\ [S(m), S(n)] &= 0, \qquad m \neq n-1, n+1, \\ \{S(m), S(n)\} &= 0, \qquad m = n-1, n+1. \end{aligned} \quad (5)$$

In words, they square to one, anticommute if they share one common vertex and commute otherwise. Analogous relations hold also with S replaced by $\tilde{S}$ in the above. Furthermore, S and $\tilde{S}$ commute with each other,

$$[S(m), \tilde{S}(n)] = 0. \quad (6)$$

It can be shown that all relations in the algebra generated by S and $\tilde{S}$ operators follow from these already listed. Furthermore, this algebra has only two irreducible representations, corresponding to two possible values of fermionic parity. Therefore, in order to perform bosonization, it is sufficient to construct operators obeying relations in Eqs. (5) and (6) in terms of spin operators. One such representation reads

$$S(n) = \sigma^1(n)\sigma^2(n+1), \qquad \tilde{S}(n) = -\sigma^2(n)\sigma^1(n+1). \quad (7)$$

Replacing operators $S(n)$ in the spin Hamiltonian by their spin representatives gives Eq. (2).

In this way, we have changed fermionic and spin variables without invoking the Jordan-Wigner transformation. This lends itself an interesting possibility that similar construction exists in higher dimensions.

Before concluding this section, we note that at the heart of the equivalence claim is the metaprinciple that systems described by the same algebras of operators are equivalent. One concrete substantiation of this, relevant for representations of Heisenberg groups, is given by the celebrated Stone–von Neumann theorem [16]. See [5,15,17] for discussion of this for algebras of fermionic bilinears, which are directly relevant for the present work.

All systems discussed in this work are defined on finite lattices. This leads to an interesting interplay between boundary conditions, conserved charges, and constraints. Explanation of these issues is one of the goals of the present paper.

In the next section, we review the spin-fermion correspondence in spatial dimension two, including the definition of constraints present in this model. In Sec. III, we explain the interplay between boundary conditions and fermionic parity. Furthermore, we solve the constraints for few small systems and check explicitly that the spectra of fermionic and spin Hamiltonians do coincide. In Sec. IV, we show that constraints can be interpreted as the condition that certain $\mathbb{Z}_2$ gauge field hidden in the bosonic theory is trivial. Modifying the form of constraints is equivalent to coupling fermions to an external gauge field. This is illustrated by a concrete calculation, in which fermions in a constant magnetic field are considered. We conclude in Sec. V and discuss a very attractive potential relation with the rapidly developing family of dualities in $(2+1)$ dimensions.

II. THE EQUIVALENT SPIN MODEL IN TWO DIMENSIONS

Generalization of the above idea to two and higher space dimensions is known for a long time [4]. In two dimensions, the fermionic Hamiltonian

$$\begin{aligned} H_f &= i\sum_{\vec{n},\vec{e}} (\phi(\vec{n})^\dagger\phi(\vec{n}+\vec{e}) - \phi(\vec{n}+\vec{e})^\dagger\phi(\vec{n})) \\ &= \frac{1}{2}\sum_l (S(l) + \tilde{S}(l)), \qquad l = (\vec{n}, \vec{e}) \end{aligned} \quad (8)$$

can be again rewritten in terms of two types of hopping operators labeled by links of a two-dimensional lattice. They obey relations which are a straightforward generalization of these from the one-dimensional case. In short, the hopping operators of the same type commute unless corresponding links have one common site. The difference is that now four, instead of two anticommuting link operators, are attached to each lattice site. Consequently, one needs bigger matrices to satisfy the corresponding algebra in higher dimensions.

In two dimensions, we choose Euclidean Dirac matrices and set (cf. Fig. 1)

$$\begin{aligned} S(\vec{n},\hat{x}) &= \Gamma^1(\vec{n})\Gamma^3(\vec{n}+\hat{x}), \qquad S(\vec{n},\hat{y}) = \Gamma^2(\vec{n})\Gamma^4(\vec{n}+\hat{y}), \\ \tilde{S}(\vec{n},\hat{x}) &= \tilde{\Gamma}^1(\vec{n})\tilde{\Gamma}^3(\vec{n}+\hat{x}), \qquad \tilde{S}(\vec{n},\hat{y}) = \tilde{\Gamma}^2(\vec{n})\tilde{\Gamma}^4(\vec{n}+\hat{y}), \\ \tilde{\Gamma}^k &= i\prod_{j\neq k}\Gamma^j. \end{aligned} \quad (9)$$

It is a straightforward exercise to show that the two-dimensional analogue of relations in Eq. (5) remains satisfied. Our Hamiltonian in the spin representation reads

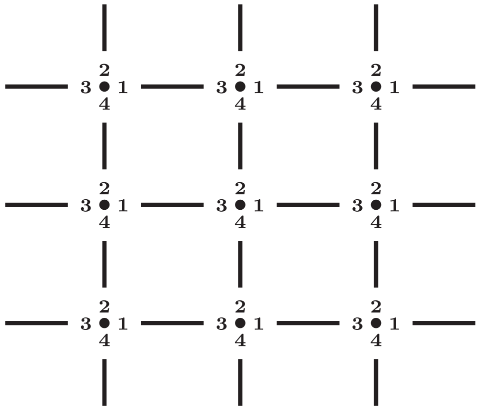

FIG. 1. Assignment of the Dirac matrices to lattice vertices—see Eq. (9).

$$H_s = \frac{1}{2}\sum_l (S(l) + \tilde{S}(l)). \tag{10}$$

Generalization to higher dimensions is simple. One needs representations of higher Clifford algebras, i.e., by larger Dirac matrices. In d dimensions, we use $2d$ anticommuting ones which corresponds to the $2d$ links meeting at one lattice site. Consequently, we have a viable candidate for a local bosonic system equivalent to free fermions in arbitrary dimensions.

The story is not over, however, since representation in Eq. (9) is redundant with respect to the fermionic one. In fact, in two space dimensions, it doubles the number of degrees of freedom per lattice site compared to the original fermionic system. Evidently, one needs additional constraints for above spins to render the exact correspondence. This can be traced to the fact that original fermionic operators S and $\tilde{S}$ obey additional relations, not present in spatial dimension one. These will have to be imposed as constraints on physical states in the spin system.

Necessary constraints are provided by the plaquette operators P_n [from now on n is a two-dimensional index $n = (n_x, n_y)$]. If we denote by C_n an elementary plaquette labeled by its lower-left corner, say, then

$$P_n = \prod_{l \in C_n} S(l). \tag{11}$$

These operators are identically 1 in the fermionic representations, while in the spin representation they merely satisfy $P_n^2 = 1$. Hence, imposing constraints

$$P_n = 1 \tag{12}$$

is necessary for the validity of the fermion-spin equivalence. It was shown already in [4] that Eq. (12) indeed correctly reduces the number of degrees of freedom per lattice site.

Details of how the claimed reduction works depend on the lattice size, boundary conditions, and other specifications. Detailed answer to this and related questions is the aim of the present work, as continued in the next sections. General explanations are given, with checks on small lattices performed analytically using symbolic algebra software [18].

III. THE CONSTRAINTS

The precise form of constraints that have to be imposed in order to make the above fermion-spin equivalence valid depends on the geometry of the lattice. To illustrate this feature, we consider two-dimensional $L_x \times L_y$ rectangular lattices [19]. Periodic or antiperiodic boundary conditions are used. Different periodicity conditions for fermions and spins are allowed,

$$\phi(n + L_x\hat{x}) = \epsilon_x \phi(n), \qquad \Gamma^k(n + L_x\hat{x}) = \epsilon'_x \Gamma^k(n),$$
$$\epsilon_x, \epsilon'_x = \pm 1, \tag{13}$$

and similarly for the other direction.

We seek to impose $\mathcal{N} = L_x L_y$ constraints from Eq. (12) to eliminate abundant degrees of freedom. However, not all of them are independent. For example, in the spin representation plaquette operators satisfy the identity

$$\prod_n P_n = 1, \tag{14}$$

which leaves at most $\mathcal{N} - 1$ independent constraints.

In addition, on finite periodic lattices, one can also construct "Polyakov line" operators

$$\mathcal{L}_x(n_y) = \prod_{n_x=1}^{L_x} S(n_x, n_y, \hat{x}),$$
$$\mathcal{L}_y(n_x) = \prod_{n_y=1}^{L_y} S(n_x, n_y, \hat{y}). \tag{15}$$

In fermionic representation, they are just pure numbers sensitive to the boundary conditions, while in spin representation their squares are unity, similarly to the plaquette operators. Hence, again they provide additional projectors. In principle, there are $L_x + L_y$ line operators, but in fact they can be shifted perpendicularly by multiplying them with appropriate rows or columns of plaquette operators [20]. Therefore, altogether there are only two more candidates for independent projectors.

It has been shown in [15] that there are no further constraints that have to be imposed besides those defined by plaquette and line operators. This has also been revisited and generalized in the work [17], which was done in parallel to this paper.

It turns out that even this set of $\mathcal{N}-1$ plaquettes and two line projectors is overcomplete. The additional structure is revealed once we consider the operator of fermion number at each site (i.e., the fermion density),

$$N(n) = \phi^\dagger(n)\phi(n). \tag{16}$$

Since Hamiltonian in Eq. (8) is moving fermions between neighboring sites only, the total number of fermions, $N = \sum_n N(n)$, is conserved, but obviously their density $N(n)$ is not.

In the spin representation, the number operator is related to the Γ^5 matrix

$$\Gamma^5(n) = \eta(-1)^{N(n)} = \eta(1-2N(n)), \tag{17}$$

where $\eta = \pm 1$ represents the freedom of defining a fermion-empty and a fermion-occupied state in the spin representation. In particular, the total fermionic parity $(-1)^N$ is given by the product of $\Gamma^5(n)$ over all lattice sites. As in the fermionic representation, N is conserved, while the number densities $N(n)$ are not. On the other hand, the plaquette and line operators do commute with the local densities. This will be exploited below when we diagonalize constraints.

Calculating directly from the definition of $\mathcal{L}$ and $(-1)^N$ operators in the spin representation, one obtains the following identity:

$$\begin{aligned}\Pi &\equiv \prod_{n_y=1}^{L_y} \mathcal{L}_x(n_y) \prod_{n_x=1}^{L_x} \mathcal{L}_y(n_x) \\ &= (-\epsilon'_x)^{L_y}(-\epsilon'_y)^{L_x}(-\eta)^{L_x L_y}(-1)^N,\end{aligned} \tag{18}$$

which [for fixed $(-1)^N$)] implies a relation between two Polyakov line projectors if at least one of L_x, L_y is odd. On the other hand, if both L_x and L_y are even, the left-hand side is insensitive to the choice of one of two values of Polyakov lines. Indeed, on the subspace defined by plaquette constraints, one has $\Pi = \mathcal{L}_x(n_y)^{L_y}\mathcal{L}_y(n_x)^{L_x}$, which is a c-number if L_x and L_y are even.

Another crucial ingredient in understanding the structure of constraints is derived by evaluating the value of Π in fermionic representation. Comparing with the result, Eq. (18), one obtains the identity

$$(-1)^N = \eta^{L_x L_y}\left(-\frac{\epsilon'_x}{\epsilon_x}\right)^{L_y}\left(-\frac{\epsilon'_y}{\epsilon_y}\right)^{L_x}. \tag{19}$$

This means that for given value of η, and boundary conditions for fermions and spins, only one of the two possible values of $(-1)^N$ is realized. This means that for the other there do not exist any solutions of constraints. We remark that a formula analogous to Eq. (19) (though in general not as transparent) exists also for more general lattice geometries.

Recall that in the ordinary fermionic Fock space the dimension of the space of states for a given value of $(-1)^N$ is $2^{\mathcal{N}-1}$. On the other hand, in our generalized system without constraints imposed, this dimension is equal to $4^{\mathcal{N}-1}$, so it is too large by a factor $2^{\mathcal{N}}$. Thus, it is natural to anticipate that there should be $\mathcal{N}-1$ independent constraints, each of which reduces the Hilbert space dimension by a factor of 2.

We have already shown that at most $\mathcal{N}-1$ plaquettes are independent and that if at least one of L_x and L_y is odd, then one Polyakov line can be eliminated in favor of the other constraints. Thus, in this case, one has exactly $\mathcal{N}-1$ independent plaquettes and one independent Polyakov loop. On the other hand, if L_x and L_y are even, it is not possible to eliminate one of the line operators. Hence, it must be that only $\mathcal{N}-2$ plaquettes are independent. This is indeed the case, as will be explained in the subsection III A.

It is now known [15,17] that it is always possible in principle to find $2^{\mathcal{N}-1}$ linearly independent solutions of constraints, corresponding to $2^{\mathcal{N}-1}$ basis vectors in one half of the Fock space. Besides the restriction to a fixed value $(-1)^N$, the two systems are indeed equivalent: there exists a unitary operator between their Hilbert spaces which carries even [i.e., commuting with $(-1)^N$] operators to spin operators according to the presented prescription. In particular, any fermionic Hamiltonian, which is always even, has the same spectrum in the fermionic representation and in the spin representation.

On the other hand, explicit solutions of constraints are known only in certain special cases. In the forthcoming discussion, we will discuss how constraints can be solved, at least for small lattice sizes. Results of all these calculations, carried out using symbolic algebra software, are in accord with theoretical predictions outlined above, providing a solid check of correctness. Needless to say, development of practical ways to deal with constraints is crucial for potential applications.

A. Some explicit examples

The complete Hilbert space of our system of spins on $L_x \times L_y$ lattice has $4^{\mathcal{N}}$ dimensions, $\mathcal{N} = L_x L_y$. States are represented by configurations

$$\{i_1, i_2, \ldots, i_{\mathcal{N}}\} \tag{20}$$

of $\mathcal{N}$ Dirac indices, $i_n = 1, \ldots, 4$ with $n = 1, \ldots, \mathcal{N}$ labeling sites of the lattice. All operators are constructed from tensor products of $\mathcal{N}$-fold four-dimensional gamma matrices and the unity. We use the specific representation of Γ^k (cf. Table I), any other equivalent choice is possible. In principle, they require $(4^{\mathcal{N}})^2$ units of computer storage; however, in general, they are sparse matrices and take only

TABLE I. Explicit representation of Euclidean Dirac matrices used in this section.

Γ^1	Γ^2	Γ^3	Γ^4	Γ^5
$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$O(4^{\mathcal{N}})$ memory size. Still, the memory requirement is the main limitation for this approach and restricts available sizes to ca. $\mathcal{N} \sim 16$.

To reduce further the memory demand, we split the whole Hilbert space into $\mathcal{N}+1$ sectors of the fixed fermion multiplicity (eigenvalue of N) $p = 0, 1, \ldots, \mathcal{N}$. In the fermionic representation, the total number of fermions is obviously conserved. The same is true in our spin representation. Namely, the corresponding number operator

$$N = \sum_n \frac{1}{2}(1 - \eta\Gamma^5(n)) \tag{21}$$

commutes with the Hamiltonian in Eq. (10). Moreover, it also commutes with all plaquette and line operators. This allows to carry out the analysis of constraints in the sectors of fixed p independently. Choosing the sector of fixed multiplicity amounts to restricting the full basis to states in Eq. (20) with $\mathcal{N} - p$ indices i in the "vacuum class," i.e., $i = 2$ or 3; then remaining p indices i' are in the "excitation class," $i' = 1$ or 4.

In practical terms, we will now be dealing with the $\mathcal{N}+1$ fixed multiplicity sectors of the full Hilbert space separately, the size of each sector being

$$2^{\mathcal{N}}\binom{\mathcal{N}}{p} \to \binom{\mathcal{N}}{p} \tag{22}$$

before and after imposing constraints on spins.

Moreover, constraint operators commute not only with the number operator N but also with each of the individual densities $N(n)$. This allows to further split the problem by performing the reduction of Hilbert space in each subsector of fixed p *and* fixed positions of p spin excitations $r_1, r_2, \ldots, r_p$ (or equivalently, fermionic coordinates) in the configuration space. Now, the reduction of dimension takes the form

$$2^{\mathcal{N}} \to 1. \tag{23}$$

Restriction to subspaces with fixed eigenvalues of $N(n)$ allows to save computer memory. Furthermore, solutions of constraints obtained this way have clear physical interpretation, as they are parametrized by space coordinates of p fermions. This is valid for all lattice sizes. It should be noted, however, that reduction in Eq. (23) is possible only for the purpose of studying the constraints. The reduced spin Hamiltonian has to be calculated in the bigger subspace of fixed p. The basis of this subspace, consisting of $\binom{\mathcal{N}}{p}$ vectors, is obtained by performing the reduction in Eq. (23) separately for each of $\binom{\mathcal{N}}{p}$ possible density configurations. This provides an appropriate basis of *constraint-satisfying* spin excitations in the larger sector of fixed fermionic multiplicity p.

To proceed further, we define the projection operators associated with all plaquettes and two Polyakov lines,

$$\Sigma_n = \frac{1}{2}(1 + P_n), \qquad \Sigma_Z = \frac{1}{2}(1 + \mathcal{L}_Z), \qquad Z = x, y, \tag{24}$$

and calculate their matrix representations, at fixed total multiplicity p. For illustration, we explicitly display below traces of successive products of all relevant projectors on 3×3 and 4×4 lattices.

For the 3×3 lattice (Table II), the reduction was performed in sectors of fixed fermion multiplicity p and proceeds according to the scheme from Eq. (22). Indeed, including successive projectors reduces dimensions by half, as expected. The last (here Σ_{33}) plaquette projector does not further reduce the dimension, in agreement with the earlier discussion. Moreover, the final result is nonzero only for multiplicities which satisfy Eq. (19). Finally, the second Polyakov line is dependent on other projectors, as is Σ_{33}, for allowed multiplicities, while it is incompatible with the rest for forbidden values of p. The final dimensionalities of the fully reduced spin Hilbert spaces agree with the sizes of the corresponding sectors with p indistinguishable fermions [see Eq. (22)], as it should be.

In the 4×4 case, the reduction was done in subsectors of fixed p fermionic coordinates [scheme in Eq. (23)]. Each of these has the same dimension $2^{\mathcal{N}}$, independently of p. As in the previous case, adding subsequent plaquette projectors reduces the size by half until one reaches the last two plaquettes. Interestingly, neither of these further reduces the remaining Hilbert space. This means that for 4×4 lattices (and more generally for $(\text{even}) \times (\text{even})$ ones) two plaquettes are dependent. This is easy to explain: for even-by-even lattices, one can split all plaquettes into two classes, according to the value of $(-1)^{n_x+n_y}$, where $n = (n_x, n_y)$ is the coordinate of the lower-left corner of the plaquette.

TABLE II. Reduction of the spin Hilbert space for 3×3 lattice in p-particle sectors. Periodic boundary conditions are assumed.

p =	0	1	2	3	4	5	6	7	8	9
$\mathrm{Tr}\,\Sigma_{11}$	256	2304	9216	21504	32256	32256	21504	9216	2304	256
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}$	128	1152	4608	10752	16128	16128	10752	4608	1152	128
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\Sigma_{13}$	64	576	2304	5376	8064	8064	5376	2304	576	64
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{21}$	32	288	1152	2688	4032	4032	2688	1152	288	32
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{22}$	16	144	576	1344	2016	2016	1344	576	144	16
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{23}$	8	72	288	672	1008	1008	672	288	72	8
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{31}$	4	36	144	336	504	504	336	144	36	4
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{32}$	2	18	72	168	252	252	168	72	18	2
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{33}$	2	18	72	168	252	252	168	72	18	2
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{x}$	1	9	36	84	126	126	84	36	9	1
$\mathrm{Tr}\,\Sigma_{11}\Sigma_{12}\ldots\Sigma_{y}$	0	9	0	84	0	126	0	36	0	1

Then, for each of the two groups independently, one has the relation

$$\prod_n P_n = (-1)^N, \qquad n_x + n_y \text{ even or odd.} \tag{25}$$

Consequently, on $(\text{even}) \times (\text{even})$ lattices, two plaquette projectors can be expressed in terms of the other. This explains the content of Table III.

On the other hand, both Polyakov line projectors are now independent. This has been explained in the discussion below Eq. (18). Regardless of parities of L_x and L_y, the number of independent projectors is $\mathcal{N}$, although they are distributed in a different way between plaquette and line operators.

The whole discussion can be repeated for other situations as well. The results are summarized in Table IV for all four cases.

The final consistency check is to calculate the spectrum of the spin Hamiltonian in the subspace defined by the constraints. Using methods outlined above, we construct for each p a basis of states satisfying all constraints. For small lattices considered in this example (see also the next section), all eigenvectors of combined projectors are analytically generated by *Mathematica* [18]. Having done that, matrix elements of the reduced spin Hamiltonian in the relevant subspace can be calculated. This exercise has been repeated for several multiplicity sectors on above lattices. In each of the considered cases, the complete spectrum of

TABLE III. Reduction of the spin Hilbert space for subsectors $0 \le p \le 16$, and fixed coordinates, on a 4×4 lattice. Sites of the lattice are ordered lexicographically, thus, e.g., sites from #1 to #5 means sites (1,1), (2,1), (3,1), (4,1), and (1,2).

Sector (p)		Even, $0 \le p \le 16$		Odd, $0 < p < 16$
Occupied sites			From # 1 to # p	
Hilbert space reduction	$\mathrm{Tr}\ \Sigma_{11}$		32768	
	$\mathrm{Tr}\ \Sigma_{11}\Sigma_{21}$		16384	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{31}$		8192	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{41}$		4096	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{12}$		2048	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{22}$		1024	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{32}$		512	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{42}$		256	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{13}$		128	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{23}$		64	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{33}$		32	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{43}$		16	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{14}$		8	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{24}$		4	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{x}$		2	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{y}$		1	
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{34}$	1		0
	$\mathrm{Tr}\ \Sigma_{11}\ldots\Sigma_{44}$	1		0

TABLE IV. Number of independent projectors and consistent multiplicities for periodic boundary conditions in both representations, $\epsilon = \epsilon' = 1$.

L_x	L_y	Plaquettes	Lines	Multiplicity
Odd	Odd	$\mathcal{N} - 1$	$\mathcal{L}_x$ or $\mathcal{L}_y$	Odd
Odd	Even	$\mathcal{N} - 1$	$\mathcal{L}_x$	Odd
Even	Odd	$\mathcal{N} - 1$	$\mathcal{L}_y$	Odd
Even	Even	$\mathcal{N} - 2$	$\mathcal{L}_x$ and $\mathcal{L}_y$	Even

known eigenenergies of p free fermions was analytically reproduced.

IV. MODIFIED CONSTRAINTS AND BACKGROUND FIELDS

Above discussion addressed solely the case where all plaquette operators were constrained to unity. In principle, however, one could consider the whole family of $2^{\mathcal{N}}$ modified constraints

$$P_n = \pm 1, \qquad 1 \leqslant n \leqslant \mathcal{N}. \tag{26}$$

Such sectors exist in the unconstrained spin system, which raises the question of their interpretation. The answer is simple and instructive, as will be discussed now.

Consider the following modification of the original fermionic Hamiltonian in Eq. (1):

$$\begin{aligned} H_f &= i\sum_{\vec{n},\vec{e}}(U(\vec{n},\vec{n}+\vec{e})\phi(\vec{n})^\dagger\phi(\vec{n}+\vec{e}) \\ &\quad - U(\vec{n},\vec{n}+\vec{e})\phi(\vec{n}+\vec{e})^\dagger\phi(\vec{n})) \\ &= \frac{1}{2}\sum_l(U(l)S(l) + U(l)\tilde{S}(l)), \end{aligned} \tag{27}$$

where $U(l)$ is an additional $\mathbb{Z}_2$ field assigned to links l. Then in the spin representation

$$H_s = \frac{1}{2}\sum_l(U(l)S(l) + U(l)\tilde{S}(l)), \tag{28}$$

with the same variables $U(l)$, and $S(l)$ given by Eq. (9). Clearly, these Hamiltonians describe fermions and/or corresponding spins in a background $\mathbb{Z}_2$ gauge field. As for the free Hamiltonian (and more generally any Hamiltonian), systems described by H_f and H_s are equivalent, as long as we restrict the spin Hilbert space in a way discussed in the previous section. We note in passing that this provides an extension of the fermion-spin equivalence to the case of external fields as well.

Interestingly, it is also possible to introduce the background gauge field in a way that it is not explicitly visible in the spin Hamiltonian [21]. Indeed, one can absorb the $U(l)$ factors into new hopping operators [22] and define

$$S'(l) = U(l)S(l), \qquad \tilde{S}'(l) = U(l)\tilde{S}(l). \tag{29}$$

This does not change the commutation rules obeyed by these operators. Now, the spin Hamiltonian does not explicitly depend on the external field,

$$H'_s = \frac{1}{2}\sum_l(S'(l) + \tilde{S}'(l)), \tag{30}$$

but the constraints on the new spin variables do. They readily follow from Eq. (11),

$$P'_n = \prod_{l\in C_n} U(l). \tag{31}$$

That is, the system of new spins is not free, but remembers the interactions via constraints in Eq. (31) only. In other words, there are two ways of introducing minimal interaction with the external field which are as follows:

(1) By introducing link variables explicitly into the Hamiltonian and imposing the "free" form of the constraint in Eq. (12).
(2) By using the free spin Hamiltonian from Eq. (10) with "interacting" constraint in Eq. (31).

We emphasize that the first method is viable for any interactions, because the equivalence between fermions and spins is valid for any Hamiltonian. The second method is possible due to the specific structure of the minimal coupling, which amounts to introducing parallel transports in any term in the Hamiltonian which involves products of on distinct lattice sites charged under the gauged symmetry. It provides an interesting interpretation of the whole spin Hilbert space.

On the fermionic side, the Hamiltonian in Eq. (27) is that of two-dimensional fermions in the fixed, external gauge field of the Wegner type [23]. The gauge field is not dynamical. On the other hand, our spin system is also coupled to the same gauge field: various boundary conditions are probing different gauge invariant classes of the $\mathbb{Z}_2$ variables [24].

The phenomenon discussed above will be illustrated by working out a simple example in subsection IV A.

One particularly interesting feature of the presented construction is that the allowed value of $(-1)^N$ becomes dependent on the background field. More precisely, let $(-1)^{N_0}$ be the right-hand side of Eq. (19). In the presence of the field $U(l)$, relation Eq. (19) is modified to

$$(-1)^N = (-1)^{N_0}\cdot\prod_l U(l), \tag{32}$$

where the product is taken over all links of the lattice. Derivation of this formula is analogous to the case of vanishing background field. An interesting gauge-theoretic interpretation of this relation has been proposed in [17] and is briefly reviewed in Sec. V.

A. A soluble example

Consider the configuration of Wegner variables given by

$$U_x(x,y) = (-1)^y, \qquad U_y(x,y) = 1, \tag{33}$$

where we assume that L_y is even. In this case, the fermionic Hamiltonian in Eq. (27) can be diagonalized analytically [25]. The one-particle spectrum reads

$$E^{(1)}_{\mathrm{mag}}(k_x,k_y) = \pm 2\sqrt{\sin^2\left(\frac{2\pi k_x}{L_x}\right) + \sin^2\left(\frac{2\pi k_y}{L_y}\right)}, \tag{34}$$

with $1 \leqslant k_x \leqslant L_x$ and $1 \leqslant k_y \leqslant \frac{L_y}{2}$, while in the free case one has

$$E^{(1)}_{\mathrm{free}}(k_x,k_y) = 2\sin\left(\frac{2\pi k_x}{L_x}\right) + 2\sin\left(\frac{2\pi k_y}{L_y}\right), \tag{35}$$

with $1 \leqslant k_z \leqslant L_z$ and $z = x,\, y$.

Configuration given by Eq. (33) leads to a Wegner's version of a constant magnetic field,

$$P_n = -1, \qquad 1 \leqslant n \leqslant \mathcal{N}. \tag{36}$$

We have repeated the procedure outlined in Sec. III A for the 3×4 lattice in order to reproduce this result. Table V shows, familiar by now, pattern of reduction of Hilbert spaces. All proceeds as before, the new element being the distinguished role of the line projector associated with $\mathcal{L}_x$, as presented in Table IV.

Table V displays results for three different orderings (A, B, C) of applying projectors. Although the final effect is the same [26], results in the intermediate stages are different, as will be explained now. Orderings A and B differ only by the order of the two line projectors which are added at the end of the process. Before that, we employ all $\mathcal{N} = 12$ plaquette projectors. As discussed before, the last one is dependent on the rest. Then, among the two line projectors, Σ_y is ineffective, i.e., dependent on other projectors, while Σ_x is independent and reduces the remaining space, regardless of the ordering A or B of imposing the constraints.

The situation is different in the scheme C, in which line projectors are imposed before the last three plaquettes. In this case, Σ_y acts as an independent projector. This does not contradict the discussion below Eq. (18), because operators $\mathcal{L}_y(n_x)$ are independent for different n_x if not all plaquette constraints are imposed (indeed, their ratio is precisely the product of some number of plaquette operators). The total number of independent constraints is equal to $\mathcal{N}$, so two among the last three plaquette constraints in the ordering C have to be ineffective. This is indeed seen in Table V. Furthermore, the final size of the one-particle sector is the correct one.

Matrix elements of the spin Hamiltonian in the one-particle sector were calculated with two choices of constraints and boundary conditions which are as follows:

(1) Free [Eq. (12)] together with $\mathcal{L}_x(1) = 1$, $\mathcal{L}_y(1) = 1$.

(2) Magnetic [Eq. (36)] and $\mathcal{L}_x(1) = -1$, $\mathcal{L}_y(1) = 1$.

In both cases, the correct fermionic spectrum was reproduced from the reduced spin Hamiltonian.

TABLE V. Reduction of the spin Hilbert space for the 3×4 lattice in the one excitation sector, with orderings (A, B, C) of applying projectors. Periodic boundary conditions are used.

p	1				
Tr 1	49152				
$\mathrm{Tr}\Sigma_{11}$	24576				
$\mathrm{Tr}\Sigma_{11}\Sigma_{21}$	12288				
$\mathrm{Tr}\ldots\Sigma_{31}$	6144				
$\mathrm{Tr}\ldots\Sigma_{12}$	3072				
$\mathrm{Tr}\ldots\Sigma_{22}$	1536				
$\mathrm{Tr}\ldots\Sigma_{32}$	768				
$\mathrm{Tr}\ldots\Sigma_{13}$	348				
$\mathrm{Tr}\ldots\Sigma_{23}$	192				
$\mathrm{Tr}\ldots\Sigma_{33}$	96				
$\mathrm{Tr}\ldots\Sigma_{14}$	48	$\mathrm{Tr}\ldots\Sigma_{14}$	48	$\mathrm{Tr}\ldots\Sigma_{x}$	48
$\mathrm{Tr}\ldots\Sigma_{24}$	24	$\mathrm{Tr}\ldots\Sigma_{24}$	24	$\mathrm{Tr}\ldots\Sigma_{y}$	24
$\mathrm{Tr}\ldots\Sigma_{34}$	24	$\mathrm{Tr}\ldots\Sigma_{34}$	24	$\mathrm{Tr}\ldots\Sigma_{14}$	12
$\mathrm{Tr}\ldots\Sigma_{x}$	12	$\mathrm{Tr}\ldots\Sigma_{y}$	24	$\mathrm{Tr}\ldots\Sigma_{24}$	12
$\mathrm{Tr}\ldots\Sigma_{y}$	12	$\mathrm{Tr}\ldots\Sigma_{x}$	12	$\mathrm{Tr}\ldots\Sigma_{34}$	12
A		**B**		**C**	

V. SUMMARY AND OUTLINE

An old proposal for local bosonization of fermionic degrees of freedom in general dimensions was revisited. Resulting spin systems are indeed local. They are subject to additional constraints which, even though local themselves, introduce effectively long range interactions. In particular, they are sensitive to the lattice geometry and fermionic multiplicities.

In this paper, we have studied and classified this dependence in detail. The necessary reduction of spin Hilbert space was demonstrated analytically for several small lattices. A number of regularities have been found. We have provided explanations which are valid for larger systems as well. Most importantly, for a given lattice size and boundary conditions, the fermion-spin equivalence holds only in the subspace defined by one of the two possible values of the fermionic parity. In this sector, imposing all constraints resulted in reduction of the spin Hilbert space to dimension appropriate for fermions.

For the above small lattices, all relevant constraints were solved with the aid of *Mathematica*. Consequently, complete eigenbases of spin states fulfilling the constraints are known analytically. Their structure is tantalizingly simple. Explicit generalization to arbitrary lattice sizes still remains a challenge.

The second step was to calculate the spectra of proposed spin Hamiltonians, reduced to the subspace defined by constraints. In all considered cases, the well-known fermionic eigenenergies have been readily reproduced.

Afterwards, the equivalence was generalized to fermions coupled minimally to a background $\mathbb{Z}_2$ gauge field. Apart from being interesting by itself, this provided a simple and intuitive interpretation of the constraints: changing the value of constraint operators is equivalent to coupling fermions to the background field. This can be achieved without introducing the background field explicitly in the spin Hamiltonian. All constraints, conceivable for this system, split into gauge invariant classes which, are in one to one correspondence with all possible gauge orbits of the external $\mathbb{Z}_2$ field. A simple proof of this fact was given. In addition, the consistency of the whole scheme was directly checked for a particular configuration of $\mathbb{Z}_2$ variables—the Wegner's analog of a constant magnetic field. Indeed, the analytically obtained spectrum of the spin Hamiltonian, reduced to the constraint-fulfilling sector, reproduced the fermionic eigenenergies in this field.

Summarizing, the exact equivalence between lattice fermions and constrained Ising-like spins was checked for a range of small lattices in $(2+1)$ dimensions. The interplay between the constraints, lattice geometry, and boundary conditions is now fully understood and classified for all fermion multiplicities and all lattice volumes. Moreover, for above small systems, the constraints were explicitly solved leading to the direct construction of the reduced spin Hilbert spaces. From a practitioner's standpoint, this provides convincing evidence for the validity of the fermion-spin equivalence by itself, since one would generally not expect an *exact* duality to hold by accident and only for small lattice sizes [27]. Proofs of validity for quite general lattice geometries and arbitrary volumes are now available in the literature, but until now almost no practical implementations have been presented. This gap is now filled.

For simplicity, most of the discussion and our calculations concentrated on the two-dimensional case. Nevertheless, extension to higher space dimensions does not present any conceptual difficulties and in fact does not bring any qualitatively new theoretical features.

Numerous dualities between various $(2+1)$-dimensional theories have been recently discovered (for reviews and references, see, e.g., [8,9]). Building on the seminal papers of Peskin, Polyakov, and others [28–30], there was a steady growth of understanding of various phenomena [31–34]. This culminated in a dramatic increase of interest in the subject in the last few years [10,35–38]. Many new structures have been found even behind the simplest and classic by now, Kramers-Wannier duality in $(1+1)$ dimensions [9,39]. To our knowledge, however, none of the available up-to-date dualities accounts exactly for the bosonization studied in this paper. On the other hand, there are several structural similarities, which we point out below.

Since gamma matrices employed here can be represented as tensor products of two Pauli matrices, our bosonization connects free fermions to a system roughly viewed as *pairs* of Ising spins living at lattice sites. Upon imposing constraints, such a model becomes exactly equivalent to above fermions from Eq. (8). A nontrivial relation emerges between the value of conserved $\mathbb{Z}_2$ charge $(-1)^N$ on one side of the duality and boundary conditions on the other. Such phenomena occur already for dualities as those of Jordan and Wigner or Kramers and Wannier, as can be seen upon carefully keeping track of various signs and global constraints; see [40] for a detailed review.

Alternatively, the unconstrained pairs of spins with local Ising-like interactions should describe fermions interacting with a dynamical $\mathbb{Z}_2$ field. An attempt to construct such a theory was recently reported in [17].

Most of dualities mentioned above involve some dynamical gauge field A. It is often the case that this gauge field obeys a modified form of the Gauss' law [41], which involves field-dependent phase factors. This is related to the fact that the gauge field action is not exactly gauge invariant, but its gauge variation depends only on its value on the spacetime boundary, and hence can be absorbed into a redefinition of the initial and final state wave functions. Such mechanism is at work in particular in Chern-Simons theories and their version suitable for finite groups, introduced by Dijkgraaf and Witten [42]. Modification of the Gauss' law has the consequence that magnetic flux excitations become paired with electric charges. This mechanism, known as flux attachment, may lead to a transmutation of statistics, due to the presence of Aharonov-Bohm phases [10,31]. Interestingly, the $\mathbb{Z}_2$ gauge field introduced in Sec. IV does also have these properties [17,43].

Mapping presented here is an exact relation between microscopic degrees of freedom for fermions and spins, as in the Jordan-Wigner duality [9]. This is different than some of the recently proposed dualities, which connect effective theories in vicinities of RG fixed points. These are typically very difficult to establish rigorously. However, one can still make arguments based on universality, matching of symmetries and anomalies, etc.

It is an attractive possibility that results established in this paper provide a microscopic realization of one of the "web of dualities" discussed, e.g., in [8,10]. One possible candidate would be the duality between a scalar field and a fermi-gauge system described in [8]. We look forward to study some of these questions in detail.

Finally, we remark that bosonization discussed in this work can be extended to higher dimensions simply by using higher dimensional Clifford algebras. In d space dimensions, this would lead to a d-plet of Ising spins living at each lattice site and interacting with nearest-neighbor couplings. It would then be interesting to see if such a mapping has its counterpart among the recently proposed webs of dualities.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Center (Poland) Grant No No. UMO-2016/21/B/ST2/01492. Research of B. R. was also supported by Polish Ministry of Science and Higher Education Grant for Ph.D. students and young researchers No. N17/MNS/000040 awarded by Jagiellonian University, Kraków.

[1] P. Jordan and E. Wigner, Über das Paulische Äquivalenzverbot, Z. Phys. **47**, 631 (1928).
[2] Y. Nambu, A note on the eigenvalue problem in crystal statistics, Prog. Theor. Phys. **5**, 1 (1950).
[3] J. B. Kogut, An introduction to lattice gauge theory and spin systems, Rev. Mod. Phys. **51**, 659 (1979).
[4] J. Wosiek, A local representation for fermions on a lattice, Acta Phys. Pol. B **13**, 543 (1982), https://inspirehep.net/literature/169185.
[5] Y.-A. Chen, A. Kapustin, and D. Radičević, Exact bosonization in two spatial dimensions and a new class of lattice gauge theories, Ann. Phys. (Amsterdam) **393**, 234 (2018).
[6] Y.-A. Chen, Exact bosonization in arbitrary dimensions, Phys. Rev. Research **2**, 033527 (2020).
[7] D. Gaiotto and A. Kapustin, Spin TQFTs and fermionic phases of matter, Int. J. Mod. Phys. A **31**, 1645044 (2016).
[8] A. Karch and D. Tong, Particle-Vortex Duality from 3d Bosonization, Phys. Rev. X **6**, 031043 (2016).
[9] T. Senthil, D. T. Son, C. Wang, and C. Xu, Duality between $(2+1)$d quantum critical points, Phys. Rep. **827**, 1 (2019), duality between $(2+1)$d quantum critical points.
[10] N. Seiberg, T. Senthil, C. Wang, and E. Witten, A duality web in $2+1$ dimensions and condensed matter physics, Ann. Phys. (Amsterdam) **374**, 395 (2016).
[11] A. Kitaev, Anyons in an exactly solved model and beyond, Ann. Phys. (Amsterdam) **321**, 2 (2006).
[12] S. B. Bravyi and A. Y. Kitaev, Fermionic quantum computation, Ann. Phys. (Amsterdam) **298**, 210 (2002).
[13] E. Zohar and J. I. Cirac, Eliminating fermionic matter fields in lattice gauge theories, Phys. Rev. B **98**, 075119 (2018).
[14] A. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (Amsterdam) **303**, 2 (2003).
[15] A. M. Szczerba, Spins and fermions on arbitrary lattices, Commun. Math. Phys. **98**, 513 (1985).
[16] N. D. Birell and P. C. W. Davies, in *Quantum Theory for Mathematicians* (Springer, New York, 2013), pp. 279–304.
[17] A. Bochniak and B. Ruba, Bosonization based on Clifford algebras and its gauge theoretic interpretation, arXiv:2003.06905.
[18] W. R. Inc., System Modeler, Version 12.1 (Champaign, IL, 2020).
[19] We assume $L_x, L_y \geq 3$ to avoid certain pathologies.
[20] In fact, one can even deform them to products of hopping operators along not necessarily straight lines. What really matters here is their winding number.
[21] An early version of this observation was made already in Ref. [15].
[22] From the gauge theory perspective, these are the gauge covariant hopping operators.
[23] F. Wegner, Duality in generalized Ising models and phase transitions without local order parameters, J. Math. Phys. (N.Y.) **12**, 2259 (1971).
[24] E. H. Fradkin and S. H. Shenker, Phase diagrams of lattice gauge theories with Higgs fields, Phys. Rev. D **19**, 3682 (1979).
[25] One employs discrete Fourier transformation and a Bogoliubov transformation.
[26] And again consistent with the condition in Eq. (19).
[27] In other words, nothing qualitatively dramatic happens with increasing lattice size. Even signatures of such subtle phenomena as phase transitions build up gradually with increasing the volume.
[28] M. E. Peskin, Mandelstam-'t Hooft duality in abelian lattice models, Ann. Phys. (Amsterdam) **113**, 122 (1978).
[29] A. M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishers, Switzerland, 1987).
[30] C. Dasgupta and B. Halperin, Phase Transition in a Lattice Model of Superconductivity, Phys. Rev. Lett. **47**, 1556 (1981).
[31] F. Wilczek, Magnetic Flux, Angular Momentum, and Statistics, Magnetic Flux, Angular Momentum, and Statistics, Phys. Rev. Lett. **48**, 1144 (1982).
[32] A. M. Polyakov, Fermi-Bose transmutations induced by gauge fields, Mod. Phys. Lett. A **03**, 325 (1988).
[33] T. Jaroszewicz and P. Kurzepa, Spin statistics, and geometry of random walks, Ann. Phys. (N.Y.) **210**, 255 (1991).
[34] K. Kajantie, M. Laine, T. Neuhaus, A. Rajantie, and K. Rummukainen, Duality and scaling in three-dimensional scalar electrodynamics, Nucl. Phys. **B699**, 632 (2004).
[35] A. Karch, B. Robinson, and D. Tong, More Abelian dualities in $2+1$ dimensions, J. High Energy Phys. 01 (2017) 017.
[36] D. T. Son, Is the Composite Fermion a Dirac Particle?, Phys. Rev. X **5**, 031027 (2015).
[37] J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu, Exact Boson-Fermion Duality on a 3D Euclidean Lattice, Phys. Rev. Lett. **120**, 016602 (2018).
[38] O. Aharony, Baryons, monopoles and dualities in Chern-Simons-matter theories, J. High Energy Phys. 02 (2016) 093.
[39] A. Karch, D. Tong, and C. Turner, A web of 2d dualities: $\mathbf{Z}_2$ gauge fields and Arf invariants, SciPost Phys. **7**, 007 (2019).
[40] D. Radičević, Spin structures and exact dualities in low dimensions, arXiv:1809.07757v3.
[41] Here we have in mind the Hamiltonian formalism with temporal gauge.
[42] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys. **129**, 393 (1990).
[43] B. Ruba, Gamma model-bosonization and gauge theory interpretation, in *Proceedings at the Asia-Pacific Lattice Conference* (2020), https://conference-indico.kek.jp/event/113/.

Bosonization of Majorana modes and edge states

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(Dated: November 23, 2021)

We present a bosonization procedure which replaces fermions with generalized spin variables subject to local constraints. It requires that the number of Majorana modes per lattice site matches the coordination number modulo two. If this condition is not obeyed, then bosonization introduces additional fermionic excitations not present in the original model. In the case of one Majorana mode per site on a honeycomb lattice, we recover a sector of Kitaev's model. We discuss also decagonal and rectangular geometries and present bosonization of the Hubbard model. For geometries with a boundary we find that certain fermionic edge modes naturally emerge. They are of different nature than edge modes encountered in topological phases of matter. Euclidean representation for the unconstrained version of a spin system of the type arising in our construction is derived and briefly studied by computing some exact averages for small volumes.

I. INTRODUCTION

Bosonization is an old subject, which is of interests both in condensed matter and high energy physics. The Jordan-Wigner transformation [1] is one of the most famous methods. It provides an effective bosonization procedure in $(1+1)$-dimensions. Its non-local character in higher dimensions leads to the search of alternative methods. There exists a zoo of proposals [2–15], including approaches motivated by the Tomonaga-Luttinger model [16], generalizations or modifications of Witten's non-abelian bosonization [17], as well as purely algebraic approaches [18]. Bosonization is also closely related to the subject of dualities, such as the Kramers-Wannier duality [19, 20] or the more recent web of dualities [11, 21–23].

Besides classical applications such as solving certain many body quantum models exactly [24] or overcoming sign problems in Monte Carlo studies [25, 26], bosonization has been invoked in the study of problems in quantum computation [27, 28] and topological phases of matter [29–33]. In [34] a two-dimensional quantum spin liquid model integrable using bosonization methods has been proposed. More recently, certain bosonization techniques were used to study inhomogeneous Luttinger liquids [35], quantum phase diagram in one-dimensional superconductors [36] and also the fractional quantum Hall fluids [37].

In [4] a bosonization technique, here referred to as the Γ model, was proposed. It transforms in a local way a fermionic model into a generalized spin system subject to constraints. This correspondence was then made more precise in [38]. Generalization and new proofs were given in [5, 39]. Constraints present in the Γ model were interpreted as the pure gauge condition for a certain $\mathbb{Z}_2$ gauge field. Modification of these constraints turned out to be equivalent to coupling fermions to an external gauge field.

The most general version of the Γ model developed so far is subject to several important limitations. First, it corresponds to a system with one fermion (hence two states) per lattice site. In this work we lift this restriction and bosonize systems with arbitrary, not necessarily even, number of Majorana modes (which are in a certain sense halves of an ordinary fermion) per lattice site. Second, in the formulations considered before the present work it was crucial that all lattice sites have an even number of neighbours. This covers many interesting examples, inlcuding the toroidal geometries frequently used in lattice simulations. Nevertheless, already for finite square lattices (say, with open boundary conditions) there exists an issue related to the existence of the boundaries. As remarked in [5, Appendix B], coordination number changes for vertices on the boundary, which may call for an adjustment of the bosonization procedure. It was argued that some Majorana modes may be present on the boundary. In this paper we come back to this issue and discuss it in detail.

We emphasize that the notion of a Majorana fermion used here has almost nothing to do with the one from high energy physics [40], which refers to a spinor field invariant under charge conjugation transformation. In particular Lorentz symmetry (or lack thereof) plays no role. Here Majorana fermions are self-adjoint operators obeying canonical anticommutation relations. Every standard fermion may be decomposed into a pair of Majoranas (real and imaginary part) in a canonical way; on the other hand pairing of Majoranas into usual fermions depends on a choice of additional structure in the space of Majorana modes [41].

One source of interest in Majorana modes in physics comes from the theory of superconductivity [42, 43]. In the presence of Abrikosov vortices [44] there may exist a finite number of Majorana zero modes per vortex, which resemble properties of Majorana particles [45]. Such phenomenon exists for eample for chiral two-dimensional p-wave superconductors [46]. Analogous vortex-related modes can be also found in superfluid ^{3}He [47]. Majorana zero modes are expected to appear also in the Moore–Read quantum Hall state [48] with filling fraction

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$\nu = \frac{5}{2}$ (the so-called Pfaffian state). Quite generally, free fermion systems characterized by nonzero $\mathbb{Z}_2$ topological invariant, such as the Kitaev's quantum wires [49], are expected to host Majorana zero modes on the boundary. Another exciting features of Majorana modes is their potential in topological quantum computation [27, 50], related to the possibility to realize non-abelian anyons. Feasibility of such topological quantum computation is still being investigated [51].

The main idea underlying the Γ model is to construct a representation of the even subalgebra of fermionic operators (i.e. the subalgebra generated by all bilinears) in terms of "spins" of sufficiently high dimension, or more precisely in terms of Euclidean Γ matrices (satisfying anticommutations relations on-site, but otherwise commuting). As observed in [4], hopping operators for fermions may be constructed given one $\Gamma(x, e)$ matrix per lattice site x for every edge e incident to the given site. In addition, one has to impose a certain constraints on states on the spin side. To represent the standard algebra of fermions one has to specify, besides hopping operators, also the fermionic parity operator on each site x. This operator has to square to 1 and anticommute with hopping operators along all edges incident to x. In other words, one needs an additional Γ matrix. If x has an even number of neighbours, this additional Γ matrix may be obtained (up to a trivial phase factor) simply by taking the product of all $\Gamma(x, e)$ with fixed x. We emphasize that this construction does not work if x has an odd number of neighbours, since then the product of all $\Gamma(x, e)$ commutes, rather than anticommutes with individual $\Gamma(x, e)$. On the other hand introducing the additional Γ matrix as an independent object would lead to spurious degrees of freedom. Hence in this version one restricts to even coordination numbers.

Generalization presented in this paper is based on a few simple observations. First, in a system featuring an odd number of Majorana modes per lattice site the on-site parity operators do not exist. Therefore the Γ model on a lattice with sites of odd degree should feature unpaired Majorana modes. Second, in presence of multiple fermionic modes per site (say, due to spin or orbital degeneracy) there exist additional independent bilinear operators which can still be bosonized if one further increases the number of Γ matrices per lattice site. In this way one obtains a mapping between fermions and spins with only one requirement: the number of Majorana modes per site x should be congruent modulo two to the number of neighbours of x. Even this condition can be eventually lifted. Indeed, if it is not satisfied, one can identify operators corresponding to spurious degrees of freedom and choose for them trivial dynamics decoupled from the rest of the system.

It turns out that the Γ model, in particular its version for arbitrary lattices proposed in this paper, shares some features with models considered in [52, 53] and [54], despite the fact that its origin and motivation were different. We will now present a short comparison between these models. In [52] the vector exchange model defined in terms of bond algebra was proposed. Similarly like in our case and the models discussed by Kapustin et al. [2, 3], the main idea was to say that two models (i.e. the fermionic and bosonic ones) are equivalent if and only if their operator algebras are isomorphic. That is, the statement was purely kinematic and Hamiltonian-independent. The idea of using higher dimensional representations of Clifford algebras instead of the Pauli matrices was introduced in [52] in order to define higher spin (e.g. $\frac{3}{2}$, $\frac{7}{2}$ etc.) analogues of the Kitaev's model. To proceed with such fermionization procedure the need for lattices of coordination number different than 3 emerged. The interplay between the dimension of the representation and the valency of lattice vertices is also discussed therein. Relation between constraints and the choice of a $\mathbb{Z}_2$ gauge field is also discussed and the counting of degrees of freedom is performed. In contrast, we start with the fermionic theory and perform the bosonization procedure based on the modification of the original Γ model. The (sector of) higher spin Kitaev's model is a result of this procedure. We also allow for multiple Majorana modes on different sites and this number may in principle vary from site to site. As a consequence of the general bosonization procedure, the relation of "unpaired" Majorana modes and generators of the Clifford algebra associated to vertices is established. The fermionization method analogous to the one in [52] was also proposed, at the same time, in [53] and [54]. In the former case the periodic boundary conditions were assumed, so that the role of the analogues of Polyakov lines discussed also in details in [39] began to be important. The role of constraints was discussed, together with the flux-attachment mechanism [55] and the interpretation of modifying the constraints as a coupling to some external $\mathbb{Z}_2$ fields. We also remark that it was argued in [53] that models with a $\mathbb{Z}_2$ gauge field chosen as in [5, Appendix B] may play a role for p-wave superconductors. The discussion at the beginning of [54] is in the same spirit as in [53]. In [54] the bulk-boundary correspondence is discussed in more detail for such models. As pointed out in [52–54], these so-called Γ-matrix models may have applications for spin liquids, (3+1)-dimensional topological insulators and the B-phase of ^{3}He.

The organization of the paper is as follows. Details of our construction are presented in Section II. Then in Section III we present examples: relation to Kitaev's model on hexagonal lattice, bosonization on a decagonal lattice and bosonization of the Hubbard model on a rectangular lattice. Afterward, in Section IV, we discuss boundary effects in the Γ model. We describe the example of square lattice with open boundary conditions and compare edge modes identified there with those arising on the boundary of some topological phases of matter.

In Section V an Euclidean representation of the simplest, unconstrained Γ model on a regular honeycomb lattice is proposed and briefly studied. The time evolution generated by spin Hamiltonians considered here is more complicated than in the standard Ising-like cases. Accordingly, Euclidean three-dimensional spin systems emerging in this Section are unknown and interesting by themselves. The feasibility of the standard, intermediate-volume, Monte Carlo studies is crudely assessed on the basis of the exact small-volume calculations.

II. THE BOSONIZATION METHOD

We consider a lattice system with fermionic degrees of freedom, whose number may vary from site to site. Real (Majorana) fermionic operators on the lattice site x will be denoted by $\psi_\alpha(x)$, with the index α (labeling Majorana modes) running from 0 to $n(x)$ with $n(x) \geq 0$. They are hermitian and satisfy anticommutation relations

$$\psi_\alpha(x)\psi_\beta(y) + \psi_\beta(y)\psi_\alpha(x) = 2\delta_{x,y}\delta_{\alpha,\beta}. \tag{1}$$

The total number of independent Majorana operators

$$n = \sum_x (n(x) + 1) \tag{2}$$

is assumed to be even, which guarantees that

$$(-1)^F = i^{\frac{n}{2}} \prod_x \prod_{\alpha=0}^{n(x)} \psi_\alpha(x) \tag{3}$$

anticommutes with every fermion. We assume that $(-1)^F$ is an exactly conserved quantity. Otherwise the Hamiltonian may be arbitrary.

We will be interested in the algebra of even operators (i.e. operators commuting with $(-1)^F$). Any even operator may be expressed as a linear combination of products of bilinears of the following two types:

$$S(e) = \psi_0(x)\psi_0(y) \text{ for an edge } e \text{ from } x \text{ to } y, \tag{4a}$$
$$T_\alpha(x) = \psi_0(x)\psi_\alpha(x) \text{ for } \alpha \neq 0. \tag{4b}$$

All S and T operators are skew-hermitian and square to -1. Furthermore we have that

- $S(e)S(e') = \pm S(e')S(e)$, with the minus sign only if e shares exactly one endpoint with e',
- $S(e)T_\alpha(x) = \pm T_\alpha(x)S(e)$, with the minus sign only if x is incident to e,
- $T_\alpha(x)T_\beta(y) = \pm T_\beta(y)T_\alpha(x)$, with the minus sign only if $x = y$ and $\alpha \neq \beta$.

It can be shown [5] that all relations in the algebra of even operators are generated by those given above and what will be called loop relations: if edges $e_1, \ldots, e_m$ form a loop (i.e. e_i terminates at the initial point of e_{i+1}, with the convention that $e_{m+1} = e_1$), then

$$S(e_1) \ldots S(e_m) = 1. \tag{5}$$

To bosonize the system, we generalize the approach proposed in [5]. For each lattice site x we construct a Clifford algebra with generators $\Gamma(x, e)$, one for each edge e incident to x, and $\Gamma'_\alpha(x)$ with $0 < \alpha \leq n(x)$. They are hermitian matrices satisfying

$$\Gamma(x, e)\Gamma(x, e') + \Gamma(x, e')\Gamma(x, e) = 2\delta_{e,e'}, \tag{6a}$$
$$\Gamma'_\alpha(x)\Gamma'_\beta(x) + \Gamma'_\beta(x)\Gamma'_\alpha(x) = 2\delta_{\alpha,\beta}, \tag{6b}$$
$$\Gamma(x, e)\Gamma'_\alpha(x) + \Gamma'_\alpha(x)\Gamma(x, e) = 0. \tag{6c}$$

Gamma matrices located on distinct lattice sites are taken to commute, and the full Hilbert space is the tensor product of on-site Hilbert spaces. In this sense the new system is bosonic. Fermionic bilinears are mapped to bosonic operators according to the local prescription

$$\widehat{S}(e) = i\,\Gamma(x, e)\Gamma(y, e) \text{ for an edge } e \text{ from } x \text{ to } y, \tag{7a}$$
$$\widehat{T}_\alpha(x) = i\,\Gamma'_\alpha(x) \text{ for } \alpha \neq 0, \tag{7b}$$

where the hat serves as an indicator that we are referring to the bosonized operators, rather than those in the original fermionic system. It is straightforward to check that $\widehat{S}$ and $\widehat{T}$ operators satisfy all relations obeyed by S and T, except for the loop relations. Instead, for a loop ℓ formed by edges $e_1, \ldots, e_m$ one has that the operator

$$W(\ell) = \widehat{S}(e_1) \ldots \widehat{S}(e_m) \tag{8}$$

squares to 1 and commutes with all $\widehat{S}$ and $\widehat{T}$. We are forced to impose the constraint

$$W(\ell)|\text{phys}\rangle = |\text{phys}\rangle \text{ for every loop } \ell. \tag{9}$$

We remark that modifying the constraint to the form

$$W(\ell)|\text{phys}\rangle = \omega(\ell)|\text{phys}\rangle \tag{10}$$

with prescribed $\omega(\ell) = \pm 1$ is equivalent [5] to coupling fermions to a background $\mathbb{Z}_2$ gauge field for the $(-1)^F$ symmetry, such that $\omega(\ell)$ is the holonomy along ℓ.

Now let $\deg(x)$ be the number of neighbors of a site x and put $N(x) = \deg(x) + n(x)$. We consider the operator

$$\gamma(x) = i^{\frac{N(x)(N(x)-1)}{2}} \prod_e \Gamma(x, e) \prod_{\alpha \neq 0} \Gamma'_\alpha(x). \tag{11}$$

Its phase is chosen so that $\gamma(x)^2 = 1$. If $N(x)$ is odd, $\gamma(x)$ commutes with all gamma matrices, so one may impose a relation $\gamma(x) = 1$ or $\gamma(x) = -1$. This amounts to choosing one of two irreducible representation of the Clifford algebra on x. If $N(x)$ is even, $\gamma(x)$ anticommutes with all gamma matrices, so it defines an additional gamma matrix: $\gamma(x) = \Gamma'_{n(x)+1}(x)$. In this case Eqs. (7) provide a bosonization of a system featuring one more Majorana fermion on the site x than we have had originally. Therefore formally we bosonize only systems with all $N(x)$ odd, but the case in which this condition is not satisfied may be handled by choosing for the spurious fermions a trivially gapped Hamiltonian, not interacting with the original fermions.

Bosonic system with constraints imposed is equivalent to the sector of the fermionic system (possibly including the spurious fermions) characterized by one of the two possible values of $(-1)^F$, defined including the spurious fermions. Which possibility is realized depends on the lattice geometry and the choice of values of γ operators. In the remainder of this section we sketch the proof of this fact, while details have been given in [5] in a slightly less general context.

First, on the space of solutions of the constraints all relations satisfied by S and T operators are obeyed by $\widehat{S}$ and $\widehat{T}$. Therefore this space is a representation of the algebra of even operators. Every representation of this algebra is a direct sum of irreducible representations, which are the two halves of the Fock space described by two values of $(-1)^F$. We will argue that only one of the two irreducible representations actually occurs in the decomposition and that the multiplicity is equal to one.

For the first part of the claim, it is sufficient to observe that the product of all S and T operators is proportional to the fermionic parity operator, while the product of all $\widehat{S}$ and $\widehat{T}$ is proportional to the product of all gamma matrices. The latter is proportional to 1, because for every lattice site we have $\gamma(x) = 1$ or $\gamma(x) = -1$. Combining these two results we conclude that $(-1)^F$ is represented by a scalar operator in the bosonic model. It is possible to determine whether it is equal to $+1$ or -1 by tracking phases carefully in the above argument. Details depend on the lattice geometry.

For the second part of the claim it suffices to calculate the dimension of the space of solutions of constraints. This is facilitated by considering also modified forms of constraints. Those are in one-to-one correspondence with gauge orbits of background $\mathbb{Z}_2$ gauge fields. Thus there are $2^{N_1-N_0+1}$ of them, where N_0 is the number of lattice sites and N_1 is the number of edges. One can show that spaces of solutions of modified constraints all have the same dimension by explicitly constructing unitary operators which map between them. Therefore the dimension of the space of solutions of constraints (9) is equal to the dimension of the whole Hilbert space in the bosonic model divided by $2^{N_1-N_0+1}$. Using the well-known values of dimensions of irreducible representations of Clifford algebras we find that there are $2^{\frac{n}{2}-1}$ linearly independent solutions of constraints. This number is equal to the dimension of one half of the Fock space.

III. EXAMPLES

We will now show how the general construction presented in the previous section works in specific examples. We begin with a model defined on the honeycomb lattice and discuss its relation with the Kitaev's model [34]. Then we discuss its three-dimensional deformation, the decagonal lattice. Discussion of boundary effects is postponed to Section IV.

We stress that this choice of lattices has been made mostly in order to simplify the presentation. Bosonization prescription from Sec. II is valid also on lattices of more complicated geometry: coordination numbers may vary from site to site and the translation symmetry is not necessary. We remark that our bosonization reduces in $(1+1)$-dimensions to the standard Jordan-Wigner transformation and as such can be thought of as its higher dimensional generalization.

A. Honeycomb lattice and Kitaev's model

In this subsection we present an example involving a honeycomb lattice. It is arguably the simplest two dimensional lattice with vertices of odd degree. Since our construction is sensitive only to the topology rather than geometry of the lattice, honeycomb lattice is equivalent to the brick wall lattice, see Fig. 1.

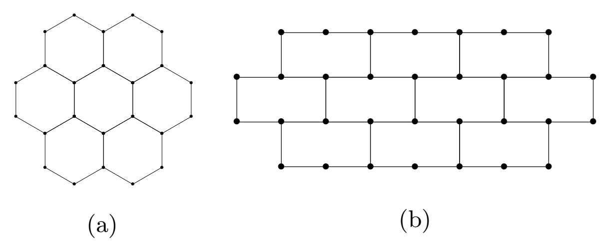


FIG. 1: (a) Honeycomb and (b) brick wall lattices are geometrically different, but topologically equivalent.

We will bosonize a system featuring one Majorana fermion ψ per lattice site. This requires three gamma matrices. They can be represented by Pauli matrices $\sigma_X, \sigma_Y, \sigma_Z$, which are assigned to edges of the lattice as illustrated in Fig. 2. Thus for a lattice site x and an edge e labeled by $I \in \{X, Y, Z\}$ we have $\Gamma(x, e) = \sigma_I(x)$.

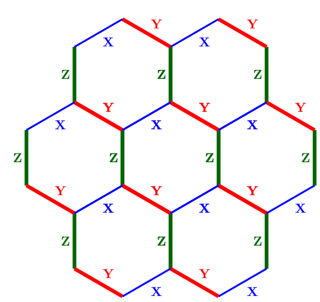


FIG. 2: The assignment of Pauli matrices.

For a plaquette P depicted in Fig. 3, the corresponding constraint takes the form $W_P|\text{phys}\rangle = -|\text{phys}\rangle$, where

$$W_P = \sigma_X(x_1)\sigma_Y(x_2)\sigma_Z(x_3)\sigma_X(x_4)\sigma_Y(x_5)\sigma_Z(x_6). \quad (12)$$

Those are the Kitaev's plaquette operators [34].

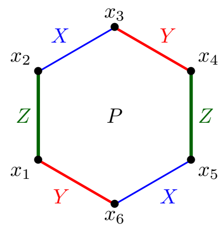


FIG. 3: A plaquette P of the honeycomb lattice.

As a specific example, let us consider the Hamiltonian

$$H = i \sum_{I \in \{X,Y,Z\}} \sum_{\substack{\text{type } I \\ \text{edges}}} J_I \, \psi(x)\psi(y), \tag{13}$$

where x, y are the endpoints of the given edge, and J_X, J_Y and J_Z are parameters of the model. According to the prescription given in Eq. (7), it corresponds to the spin Hamiltonian

$$\widehat{H} = -\sum_{I \in \{X,Y,Z\}} \sum_{\substack{\text{type } I \\ \text{edges}}} J_I \, \sigma_I(x)\sigma_I(y), \tag{14}$$

subject to the constraint $W_P = -1$ for every plaquette P. This Hamiltonian has been proposed in [34], where its study was reduced to diagonalization of quadratic fermionic Hamiltonians. Our approach provides an alternative derivation of this result. Subspaces defined by different values of W_P correspond to the Hamiltonian H modified by including a background $\mathbb{Z}_2$ gauge field.

B. Decagonal lattice

An example of a three-dimensional trivalent lattice is provided by the decagonal geometry. A convenient representation is shown in Fig. 4, where one layer of such lattice is presented, together with edges connecting it with the adjacent layers. It can be thought of as a deformation of the brick wall lattice. In the brick wall geometry, each red site was connected with a green one to its north, while in the decagonal geometry it is instead connected with a green site lying in the layer underneath.

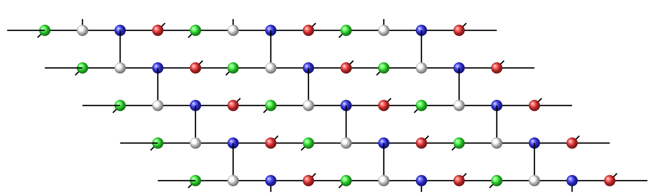
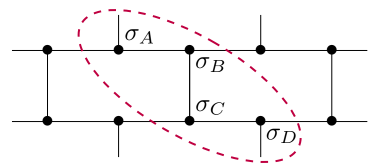
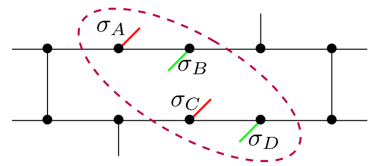


FIG. 4: One layer of the decagonal lattice.

By the similarity with the brick wall geometry, one can easily generalize the results from Section III A. Using the identification of edges between honeycomb and brick wall lattices we attach Pauli matrices to pairs (x, e) of the decagonal lattice, see Fig. 5.



(a)



(b)

FIG. 5: The assignment of Pauli matrices for (a) the brick wall lattice and (b) the decagonal lattice.

As an example, constraint associated to the plaquette from Fig. 6 takes the form $W_P|\text{phys}\rangle = -|\text{phys}\rangle$, where

$$\begin{aligned} W_P = &\sigma_X(x_1)\sigma_Y(x_2)\sigma_Z(x_3)\sigma_Z(x_4)\sigma_Z(x_5) \\ &\times \sigma_X(x_6)\sigma_Y(x_7)\sigma_Z(x_8)\sigma_Z(x_9)\sigma_Z(x_{10}). \end{aligned} \tag{15}$$

There exist also plaquettes not contained within one layer, but for the sake of brevity we will not write down the explicit formulas.

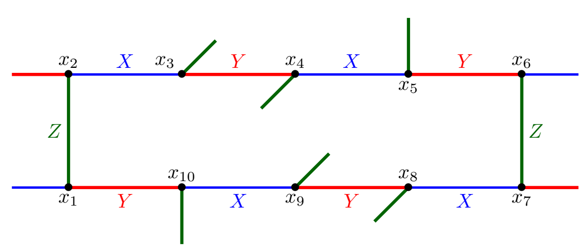


FIG. 6: Plaquette P within a single layer of the decagonal lattice.

As in the honeycomb lattice case, every edge is labeled by $I \in \{X, Y, Z\}$ and there is a correspondence between the Hamiltonians in Eqs. (13) and (14).

C. Hubbard model

In the preceding examples only one kind of fermionic variables was involved. Here we discuss the simplest model with an additional quantum number involved - the Hubbard model [56] on the square lattice.

The Hamiltonian of the Hubbard model consists of two terms, $H = H_0 + V$, where

$$H_0 = -t \sum_{\langle xy \rangle} \sum_{\sigma=\uparrow,\downarrow} \left(c_\sigma^\dagger(x) c_\sigma(y) + c_\sigma^\dagger(y) c_\sigma(x) \right), \tag{16a}$$

$$V = U \sum_x n_\uparrow(x) n_\downarrow(x), \tag{16b}$$

where the edge connecting sites x and y is denoted by $\langle xy \rangle$. Here $c_\sigma^\dagger(x)$ creates a fermion with spin σ at position x, $n_\sigma(x) = c_\sigma^\dagger(x) c_\sigma(x)$, and $t, U \in \mathbb{R}$. These fermionic operators may be decomposed into Majoranas as

$$c_\uparrow(x) = 2^{-1}(\psi_0(x) + i\psi_1(x)), \tag{17a}$$

$$c_\uparrow^\dagger(x) = 2^{-1}(\psi_0(x) - i\psi_1(x)), \tag{17b}$$

$$c_\downarrow(x) = 2^{-1}(\psi_2(x) + i\psi_3(x)), \tag{17c}$$

$$c_\downarrow^\dagger(x) = 2^{-1}(\psi_2(x) - i\psi_3(x)). \tag{17d}$$

This choice is by no means unique and we are free to (consistently) use any other relabelling of indices. After modifying accordingly the bosonization prescription, different choices will lead to equivalent bosonic models.

To perform bosonization we need seven Γ matrices per site, six of which are independent: the seventh may be taken to be the product of the first six and the imaginary unit. The bosonized Hamiltonian takes the form:

$$\widehat{H}_0 = -\frac{it}{2} \sum_{e=\langle xy \rangle} \Gamma(x,e)\Gamma(y,e) \left(\Gamma_1'(x) - \Gamma_1'(y) - i\Gamma_2'(x)\Gamma_3'(y) + i\Gamma_3'(x)\Gamma_2'(y) \right), \tag{18a}$$

$$\widehat{V} = \frac{U}{4} \sum_x \left(1 - \Gamma_1'(x)\right) \left(1 + i\Gamma_2'(x)\Gamma_3'(x)\right). \tag{18b}$$

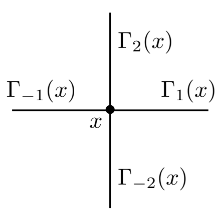


FIG. 7: The assignment of gamma matrices.

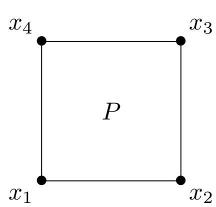


FIG. 8: A plaquette p of the rectangular lattice.

To write down the constraints, it is covenient to denote $\Gamma(x,e)$ as $\Gamma_i(x)$ for edge i pointing from x in the i-th direction, $i \in \{\pm 1, \pm 2\}$ (see Figure 7). Now consider a plaquette P as in Fig. 8. The corresponding constraint takes the form $W_P|\text{phys}\rangle = -|\text{phys}\rangle$, where

$$W_P = \Gamma_{1,2}(x_1)\Gamma_{2,-1}(x_2)\Gamma_{-1,-2}(x_3)\Gamma_{-2,1}(x_4). \tag{19}$$

Here we abbreviated $\Gamma_{i,j}(x) := \Gamma_i(x)\Gamma_j(x)$. We note that this constraint does not at all involve primed gamma matrices, which we had to introduce in order to implement multiple fermions per site. It is characteristic for the square lattice geometry.

One annoying feature of the presented construction is that spin up and spin down states are not treated completely symmetrically. Nevertheless, symmetries of the Hubbard model are implemented also in the bosonized model. First, we have conservation of the total particle number. Particle number on a single lattice site x is bosonized in the following way:

$$\sum_\sigma c_\sigma^\dagger c_\sigma \longleftrightarrow 1 - \frac{1}{2}\Gamma_1' + \frac{i}{2}\Gamma_2'\Gamma_3'. \tag{20}$$

For spin (SU(2) generators) operators we have:

$$c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow \longleftrightarrow -\frac{1}{2}\Gamma_1' - \frac{i}{2}\Gamma_2'\Gamma_3', \tag{21a}$$

$$c_\uparrow^\dagger c_\downarrow \longleftrightarrow \frac{i}{4}(1 - \Gamma_1')(\Gamma_2' + i\Gamma_3'), \tag{21b}$$

$$c_\downarrow^\dagger c_\uparrow \longleftrightarrow -\frac{i}{4}(1 + \Gamma_1')(\Gamma_2' - i\Gamma_3'). \tag{21c}$$

It noteworthy that all on-site symmetries of the original fermionic model are also on-site after bosonization. Furthermore, the corresponding charges are expressed entirely in terms of *primed* gamma matrices. This is a general property of our construction.

IV. BOUNDARY EFFECTS

A. Rectangular lattice with a boundary

We will now discuss bosonization of a system on an $L_x \times L_y$ rectangular lattice, with two Majorana fermions ψ_0, ψ_1 per lattice site. Every site x in the bulk has four neighbors, corresponding to four gamma matrices $\Gamma_{\pm i}(x)$, $i = 1, 2$, as in our discussion of the Hubbard model. According to the prescription given in Sec. II, we need also an additional gamma matrix $\Gamma_1'(x)$. It can be eliminated by imposing relations discussed below equation (11). We choose the convention $\Gamma_1'(x) = \Gamma_{-1}(x)\Gamma_1(x)\Gamma_{-2}(x)\Gamma_2(x)$. Therefore in the end we need only unprimed gamma matrices. Constraints are identical as in the discussion of the Hubbard model.

More explicitly, our bosonization prescription reads

- $i\Gamma_1(x)\Gamma_{-1}(y) \longleftrightarrow \psi_0(x)\psi_0(y)$ if y is the eastern neighbor of x,
- $i\Gamma_2(x)\Gamma_{-2}(y) \longleftrightarrow \psi_0(x)\psi_0(y)$ if y is the northern neighbor of x,
- $i\Gamma_{-1}(x)\Gamma_1(x)\Gamma_{-2}(x)\Gamma_2(x) \longleftrightarrow \psi_0(x)\psi_1(x)$.

Now, we look closely at the situation on the boundary. First, sites on the southern edge (see Fig. 9) have no neighbors in the direction -2. We may reinterpret the Γ_{-2} matrix as an additional Γ', corresponding to a spurious Majorana fermion on the boundary. More precisely, for every site x_i on the southern edge we introduce an additional Majorana operator $\chi_{\mathrm{S}}(x_i)$. Bosonization prescription for χ_S fermions takes the form

$$i\Gamma_{-2}(x_i) \longleftrightarrow \psi_0(x_i)\chi_{\mathrm{S}}(x_i). \tag{22}$$

Similarly for the northern, eastern and western edges we introduce Majorana fermions χ_{N}, χ_{E} and χ_{W}.

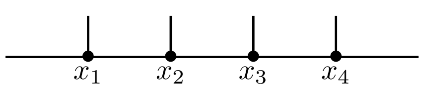


FIG. 9: The southern boundary of the rectangular lattice.

At each of the four corners (which are geometrically of codimension two) there are two χ fermions. For example the south-east corner x_{SE} hosts four Majorana operators $\psi_0(x_{\mathrm{SE}})$, $\psi_1(x_{\mathrm{SE}})$, $\chi_S(x_{\mathrm{SE}})$ and $\chi_E(x_{\mathrm{SE}})$.

We now determine the identity resulting from existence of the boundary. First, notice that for every lattice site $x = (a, b) \in \{1, \dots, L_x\} \times \{1, \dots, L_y\}$ we have

$$\prod_{a=1}^{L_x-1} \psi_0(a,b)\psi_0(a+1,b) = \psi_0(a,b)\psi_0(L_x,b). \tag{23}$$

Consequently, our bosonization prescription yields

$$\psi_0(1,b)\psi_0(L_x,b) \\ \longleftrightarrow i^{L_x-1}\Gamma_{-1}(1,b)\left(\prod_{a=1}^{L_x}\Gamma_{-1,1}(a,b)\right)\Gamma_1(L_x,b) \tag{24}$$

and

$$\psi_0(a,1)\psi_0(a,L_y) \\ \longleftrightarrow i^{L_y-1}\Gamma_{-2}(a,1)\left(\prod_{b=1}^{L_y}\Gamma_{-2,2}(a,b)\right)\Gamma_1(a,L_y), \tag{25}$$

for every $1 \le b \le L_y$ and $1 \le a \le L_x$, respectively. By a straightforward computation one can check that it results in the following correspondence

$$i^{(L_x-L_y)^2+2(L_x+L_y)}\prod_{a,b}\Gamma_{-2,2,-1,1}(a,b) \\ \longleftrightarrow \chi_{\partial_S}\chi_{\partial_N}\chi_{\partial_W}\chi_{\partial_E}, \tag{26}$$

where we have introduced the abbreviated notation $\chi_{\partial_S} = \prod_{a=1}^{L_x}\chi_S(a,1)$, and so on. Since $\Gamma_{-2,2,-1,1}(x)$ corresponds to $-i\psi_0(x)\psi_1(x)$, which is the parity operator $(-1)^{F_\psi(x)}$ for ψ fermions at site x, we end up with the following constraint

$$(-1)^{F_\psi} = \kappa\chi_{\partial_S}\chi_{\partial_N}\chi_{\partial_W}\chi_{\partial_E}, \tag{27}$$

where $\kappa = i^{-(L_x-L_y)^2+2(L_x+L_y)}$ is a geometrical phase factor. In particular, for lattices with $L_x \equiv L_y \pmod 2$ we have $\kappa = 1$.

Summarizing, we started from the system of ψ fermions, but our bosonization gave us a bosonic system equivalent to ψ fermions together with χ fermions on the boundary. Since operators corresponding on the spin side to spurious χ modes have been identified, this is not a problem. Indeed, bosonizing suitable Hamiltonian for χ fermions using formula (22) and its analogues for other components of the boundary we may make χ arbitrarily heavy, e.g. dimerized with large dissociation energy. This provides a physical interpretation for additional constraints on the boundary introduced in [5, Appendix B]. After imposing them, one obtains a spin system corresponding to ψ fermions on a lattice with boundary.

B. Comment about topological phases

One of the remarkable features of many topologically nontrivial phases of matter is their interesting (robust) behaviour on the boundaries, which are of dimension $d-1$. This is the case in particular for free fermion systems, for which one has a well-established bulk-boundary correspondence: topological invariants in the bulk signal existence of modes localized near the boundary, responsible for closing the gap in finite volume.

Robustness of the boundary modes in often interpreted as manifestation of an anomaly of the boundary theory. Presence of the anomaly implies existence of some degrees of freedom "saturating" the anomaly. On the other hand, it is also expected that the anomalous $(d-1)$-dimensional system is inconsistent on its own: it may be realized only on the boundary of a d-dimensional system. As an example, chiral fermions cannot be realized on the lattice (this is the Nielsen-Ninomiya theorem, see [57, 58]), but they may exist on the boundary or domain wall (more generally, a defect) in a higher dimensional system. This is at the heart of the bulk boundary correspondence.

On the other hand, boundary modes found in our bosonization prescription correspond to a standalone (hence "non-anomalous") system on the boundary. Now suppose that we bosonize a free fermion system with a nonzero topological invariant, say on a half-space. Then on the boundary we will have boundary modes predicted by the bulk-boundary correspondence and χ fermions described in the previous subsection. We can gap out the latter fermions by including in the Hamiltonian a suitable term localized in the boundary. This is believed not to be true for the former. In the case of invariants which remain robust in presence of interactions [59] it is natural to expect that even after including a coupling between χ fermions and ψ fermionis, boundary modes originating from a topological invariant will persist.

Of course the full picture of topological invariants and boundary modes has to involve the choice of a Hamiltonian, or at least some class of Hamiltonians. On the other hand, the discussion presented here is mostly concerned with properties of algebras of observables. It would be interesting to understand better the relation between bosonization and bulk-boundary correspondence. Such questions are relevant, for example, for the problem of discretization of chiral fermions.

V. EUCLIDEAN REPRESENTATION OF UNCONSTRAINED "MAJORANA SPINS"

The next goal is to construct an Euclidean Ising-like action, with two different couplings, β_t and β_s, which in the continuous time limit

$$\beta_t \to \infty, \quad \epsilon = e^{-\beta_t} \to 0, \quad \beta_s = \epsilon\lambda \to 0 \tag{28}$$

is described by the Hamiltonian (14) with parameters $J_X = J_Y = 1$, $J_Z = \lambda$ (we will not impose constraints at this point). Following [60, 61], this is done by demanding that the transfer matrix elements defined by the Boltzmann weight

$$\langle s'|T|s\rangle = e^{-\mathsf{L}(s',s)} \tag{29}$$

coincide with these of the Euclidean evolution operator

$$\langle s'|T|s\rangle = \langle s'|e^{-\epsilon H}|s\rangle = \langle s'|1-\epsilon H|s\rangle \tag{30}$$

up to terms of order ϵ. Here ϵ is the elementary time step and s and s' denote configurations of spins at subsequent time instants.

The time evolution generated by (14) consists of elementary double-spin flips, in contrast to the Ising system in which the dynamics is driven by single spin flips. In order to gain some orientation in this problem, we start by deriving an Euclidean action for a simpler, one-dimensional quantum Hamiltonian

$$\begin{aligned} H_{1d} = &-\sum_k \sigma_X(x_k)\sigma_X(x_{k+1}) \\ &-\lambda\sum_k \sigma_Z(x_k)\sigma_Z(x_{k+1}). \end{aligned} \tag{31}$$

A. Basic idea and the $(1+1)$-dimensional example

In the Ising model, the basic trick relating Hamiltonian and functional formulations is to classify all variations of a multiple-spin state into classes with fixed number of single spin flips.

On the Euclidean side, the number of single flips between two time slices is counted by the two-row action

$$\mathsf{L}_1(s',s) = \frac{1}{2}\sum_k (1 - s_k s'_k). \tag{32}$$

It corresponds to the Hamiltonian

$$-\sum_k \sigma_X(x_k). \tag{33}$$

In the present case (31), we are seeking to single out the *double spin flips* out of all possible changes of a row of spins. Therefore we begin with the Euclidean eight-spin action which counts *isolated* double flips

$$\begin{aligned} \mathsf{L}_2^{(8)} = \frac{1}{2^4}\sum_k &(1+s_{k-1}s'_{k-1})(1-s_k s'_k) \\ &\times (1-s_{k+1}s'_{k+1})(1+s_{k+2}s'_{k+2}), \end{aligned} \tag{34}$$

Simpler functions can be also used, hence we shall omit the "(8)" superscript if not necessary.

We need to arrange the final, Euclidean action such that in the continuous time limit it gives weight ϵ to double flips while all other, single and multiple, flips are of higher order in $\epsilon = e^{-\beta_t}$. This is achieved by the combination

$$\mathsf{L}_{\mathrm{kin}}(s',s) = \beta_t \left(p(\mathsf{L}_1 - 2\mathsf{L}_2) + \mathsf{L}_2 \right), \tag{35}$$

where $p \geq 2$ is a free parameter.

It is easy to check that $\mathsf{L}_2^{(8)}$ may also be replaced in (35) by the simpler function

$$\mathsf{L}_2^{(6)} = \frac{1}{8}\sum_k (1+s_{k-1}s'_{k-1})(1-s_k s'_k)(1-s_{k+1}s'_{k+1}). \tag{36}$$

This definition prescribes different weights to non-leading transitions, but results in the same double flip kinetic part of (31). This is an illustration of the well known fact that many different Euclidean discretizations have the same continuous time limit, hence also the same Hamiltonian.

Action for a single transition has to be supplemented by a potential term:

$$\mathsf{L}(s', s) = \mathsf{L}_{\text{kin}}(s', s) + \mathsf{L}_{\text{pot}}(s', s), \tag{37a}$$

$$\mathsf{L}_{\text{pot}}(s', s) = -\frac{\beta_s}{2} \sum_k \left(s_k s_{k+1} + s'_k s'_{k+1} \right). \tag{37b}$$

Complete Euclidean action for the $L_x \times L_t$ spins is obtained by composing elementary transfer matrices, which amounts to adding the corresponding actions:

$$S(s(L_t), \ldots, s(1)) = \sum_{t=1}^{L_t} \mathsf{L}(s(t+1), s(t)). \tag{38}$$

This concludes our construction of the two-dimensional, Euclidean system which in the continuum time limit is described by the Hamiltonian (31).

1. $\sigma_Y \sigma_Y$ terms - the phases.

The second example deals with the phase generating kinetic terms

$$\begin{aligned} H^{\text{ph}}_{1d} = &- \sum_{k \text{ even}} \sigma_X(x_k)\sigma_X(x_{k+1}) - \sum_{k \text{ odd}} \sigma_Y(x_k)\sigma_Y(x_{k+1}) \\ &- \lambda \sum_k \sigma_Z(x_k)\sigma_Z(x_{k+1}) \end{aligned} \tag{39}$$

still in one space dimension.

Begin with an evolution of a two spin system:

$$s = \{s_1, s_2\} \to s' = \{s'_1, s'_2\}. \tag{40}$$

As far as the change of spin states is considered, the action of $\sigma_Y \sigma_Y$ is the same as that of $\sigma_X \sigma_X$. The only difference is a phase factor:

$$\begin{aligned} &\sigma_Y(x_1)\sigma_Y(x_2)|s_1, s_2\rangle \\ &= \exp\left(\frac{i\pi}{2}(s_1 + s_2)\right)\sigma_X(x_1)\sigma_X(x_2)|s_1, s_2\rangle. \end{aligned} \tag{41}$$

Generalization to a whole row of L spins is straightforward. The kinetic term of the Hamiltonian (39) will be reproduced by the action (35) supplemented by a phase (41) for each odd edge. This gives for the new action of the two complete rows (and with the unchanged diagonal potential term)

$$\begin{aligned} \mathsf{L}_{\text{ph}}(s', s) = &\beta_t \left(2(S_1 - 2S_2) + S_2\right) + \beta_s S^{\text{pot}} \\ &+ \frac{i\pi}{2}\frac{1}{2^3} \sum_{x-\text{odd}} (s_x + s_{x+1})(1 + s_{x-1}s'_{x-1})(1 - s_x s'_x)(1 - s_{x+1}s'_{x+1}). \end{aligned} \tag{42}$$

B. $(2+1)$-dimensional system

As a $(2+1)$-dimensional example we consider here the honeycomb lattice discussed in III A. As remarked therein, it is convenient to represent it in a brick wall form, cf. Fig. 1.

Our Hamiltonian is of the form (14) with parameters $J_X = J_Y = 1$ and $J_Z = \lambda$, and can be written as a sum of two terms

$$H = H_{\text{kin}} + \lambda H_{\text{pot}}, \tag{43}$$

where the kinetic term contains sums over all edges of type X and Y, while the potential one is a sum over edges of type Z. The labelling of the edges of the brick wall lattice is shown explicitly in Fig. 10 and is consistent with the one in Fig. 2.

Derivation of an Euclidean action of a three-dimensional (x, y, t), periodic in all directions, system is very similar to the previous $(1+1)$-dimensional example.

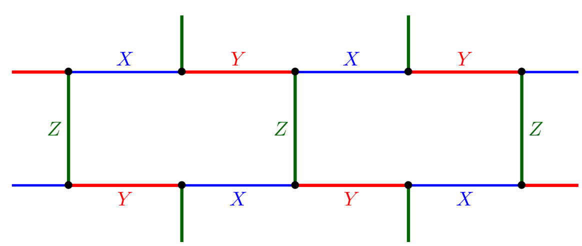


FIG. 10: The brick wall lattice with the assignment of Pauli matrices.

The two kinetic (i.e. $\sigma_X \sigma_X$ and $\sigma_Y \sigma_Y$) terms in (43) are represented by the same six- or eight-spin couplings between the adjacent time slices plus the appropriate phase, which naturally generalizes the (1+1)-dimensional phase in the last term of (42) to three Euclidean dimensions.

On the other hand diagonal, in the Hamiltonian form, potential terms are represented by the standard Ising-like, ferromagnetic couplings along the y-direction. They are located on the shorter edges of bricks at each time slice. Hence, they are staggered in accord with the (t-independent) $x-y$ parity, $\zeta_{xy} = (-1)^{x+y}$, of a site originating given Z-edge in the potential term. The final action reads

$$S_{3D} = \beta_t \sum_{x,y,t} O^{(6)}_{x,y,t} + \beta_s \sum_{\substack{x,y,t,\\ \zeta_{xy}=1}} O^{(2)}_{x,y,t} + \frac{i\pi}{2} \sum_{\substack{x,y,t,\\ \zeta_{xy}=-1}} O^{(7)}_{x,y,t}, \tag{44}$$

with the phase operator $O^{(7)}_{x,y,t}$ being the direct generalization of above $O^{(7)}_{x,t}$ to three dimensions and similarly for other couplings:

$$O^{(7)}_{x,y,t} = \frac{1}{2^3}(s_{x,y,t} + s_{x+1,y,t})(1 + s_{x-1,y,t}s_{x-1,y,t+1})(1 - s_{x,y,t}s_{x,y,t+1})(1 - s_{x+1,y,t}s_{x+1,y,t+1}), \tag{45a}$$

$$O^{(6)}_{x,y,t} = \frac{1-2p}{8}(1 + s_{x-1,y,t}s_{x-1,y,t+1})(1 - s_{x,y,t}s_{x,y,t+1})(1 - s_{x+1,y,t}s_{x+1,y,t+1}) + \frac{p}{2}(1 - s_{x,y,t}s_{x,y,t+1}), \tag{45b}$$

$$O^{(2)}_{x,y,t} = -s_{x,y,t}s_{x,y+1,t}. \tag{45c}$$

The action (44) describes then a three-dimensional Ising-like system. Together with the corresponding constraints (still to be implemented) it would provide an equivalent, Euclidean representation of a single, quantum Majorana spin on a two-dimensional spatial lattice.

Even without the constraints the system is still interesting *per se*. Its thermodynamics, the phase diagram, order parameters are unknown at the moment and could be studied with standard methods of statistical physics. Such studies would also provide, among other things, some information about the constraints themselves.

The Boltzmann factor associated with (44) is not positive. However the origin of its phases is now conceptually simple. Below we look how severe is the sign problem in these unconstrained Euclidean models.

C. The sign problem

The standard (and practically only) method to deal with non-positive weights is the reweighting [62, 63]. Instead of potentially negative Boltzmann factor $\rho = \exp(-S)$, one uses as a Monte Carlo (MC) weight its absolute value $\rho_A = |\rho|$, correcting at the same time all observables for this bias.

Whether such an approach is practical can be readily judged from the average value of a sign of the exact Boltzmann factor

$$\langle \text{sign} \rangle \equiv \left\langle \frac{\rho}{\rho_A} \right\rangle_A = \frac{\mathcal{Z}}{\mathcal{Z}_A} \tag{46}$$

averaged over the modulus ρ_A. If this average is close to 0, the method fails. If the contrary is true, say for some intermediate volumes, one may expect to obtain meaningful estimates.

We have calculated analytically above average for both $(1+1)$- and $(2+1)$-dimensional models by employing the transfer matrix technique for a range of small volumes. It is seen below that the sign problem is not very severe in this case. Consequently, MC studies remain a viable approach to explore these systems in detail.

1. $(1+1)$-*dimensions*

Partition functions $\mathcal{Z}$ and $\mathcal{Z}_A$ were calculated exactly by summing Boltzmann factors $\exp(-S_{2D})$ and $|\exp(-S_{2D})|$, as defined in Eqns (42). In Fig.11 the average sign is shown for a range of two dimensional volumes and various couplings β_t and β_s. The results are displayed as a function of a time step, $\epsilon = \exp(-\beta_t)$, and parameterized by different couplings $\lambda = \frac{\beta_s}{\epsilon}$ in the Hamiltonian (39). Second column displays analogous results for larger penalty parameter p.

The sign problem seems manageable for a sizeable part of the parameter space. It vanishes entirely for $\epsilon \to 0$.

Increasing the penalty parameter p also helps, since then some undesired transitions vanish faster with ϵ.

Both of these features show up also in our three-dimensional system. They can be readily understood and used for our advantage, as discussed below.

2. $(2+1)$-*dimensions*

For the three-dimensional Euclidean system (44) of volume $V = L_x L_y L_t$ a brute-force summation of all 2^V

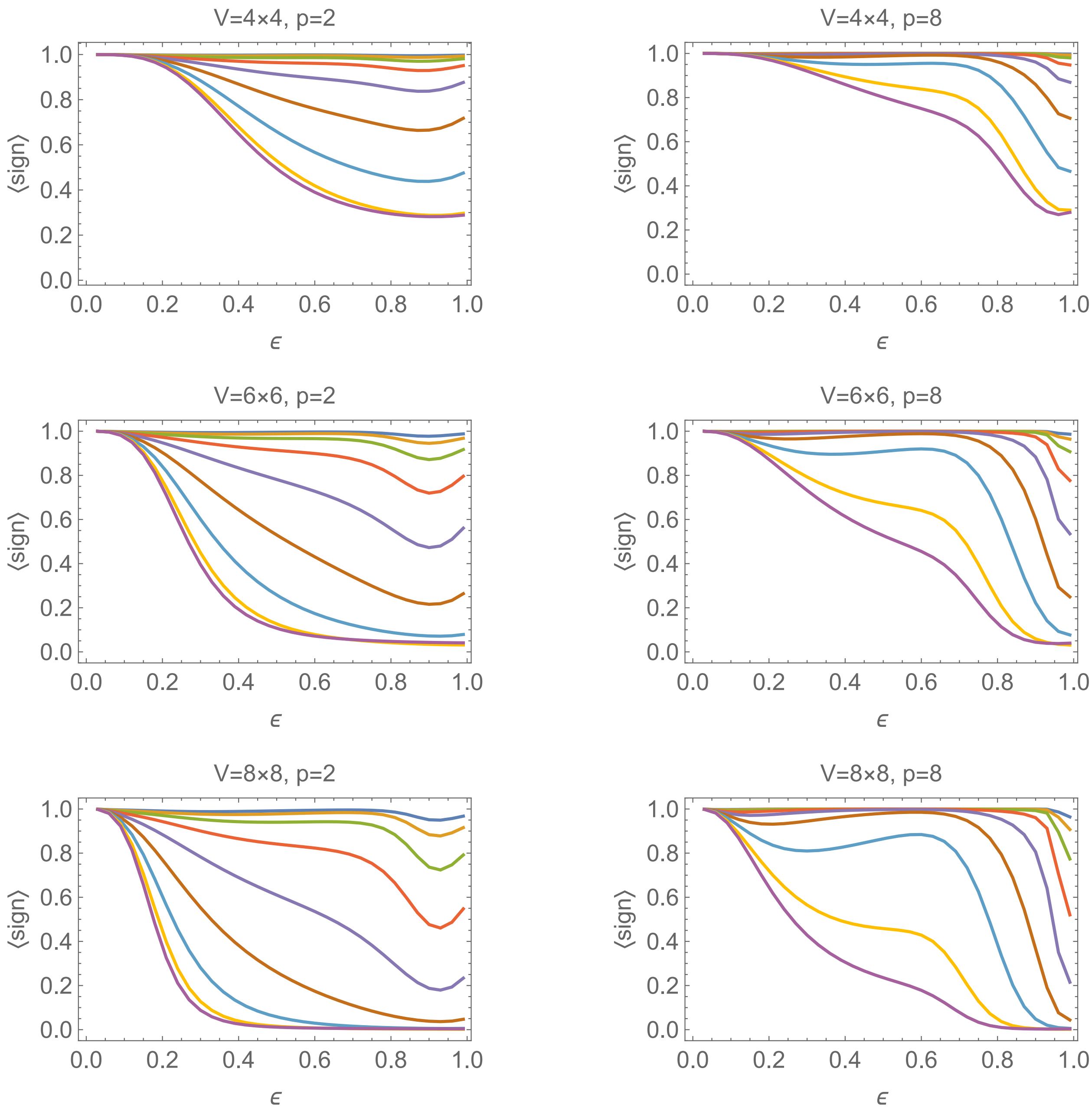


FIG. 11: Exact results for the average sign $\langle \text{sign} \rangle$ for a range of two dimensional volumes V and for the penalty parameter $p = 2$ (left column) or $p = 8$ (right column). Plots are presented for λ values (from bottom to top) 0.1, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75 and 2.

terms becomes already a challenge. Still it was possible to obtain the value of $\langle \text{sign} \rangle$ for $V = 4 \times 4 \times 3$, as shown in Fig. 12. It was done by constructing two subsequent transfer matrices in the y direction.

Again, as in the $(1+1)$-dimensions, the phase is harmless for small ϵ. This feature improves dramatically with increasing the penalty parameter.

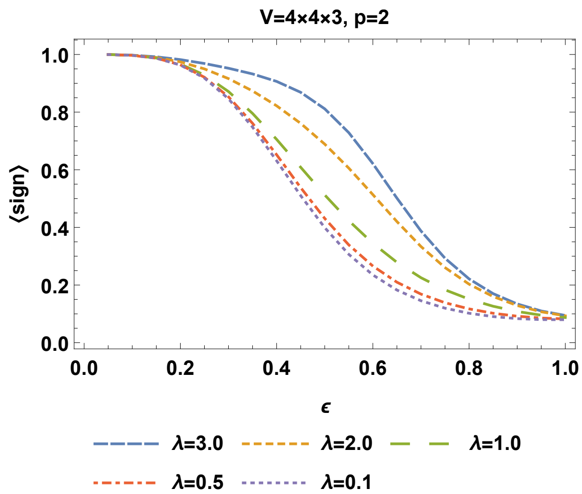
In addition, for $L_t = 2$ no phase was observed in all cases. That is $\langle \text{sign} \rangle = 1$ for all values of parameters and for all studied dimensions.

3. *The sign problem - summary*

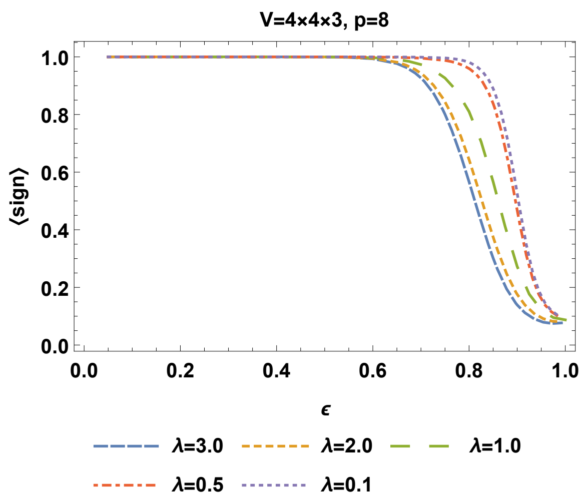
All the regularities observed above can be readily understood and generalized for arbitrary sizes of lattices, providing at the same time some guidelines for other, similar systems.

Consider first the case $L_t = 2$. The partition function

$$\mathcal{Z}^{(2)} = \sum_{s,s'} e^{-\mathsf{L}(s,s')} e^{-\mathsf{L}(s',s)} \tag{47}$$



(a)



(b)

FIG. 12: Average sign as a function of the ϵ parameter in the three-dimensional case with volume $V = 4 \times 4 \times 3$ for the penalty parameter (a) $p = 2$ and (b) $p = 8$. The range of λ parameter is explicitly given for both cases.

is the sum over two-composite states of spins at the two time slices. The non-zero phase can occur only if s and s' differ by a double flip. However in this case the phases of $e^{-\mathsf{L}(s,s')}$ and $e^{-\mathsf{L}(s',s)}$ cancel and the result is positive for each pair of configurations, as found above.

On the other hand, already for $L_t = 3$ there are three states in the game

$$\mathcal{Z}^{(3)} = \sum_{s,s',s''} e^{-\mathsf{L}(s,s'')-\mathsf{L}(s'',s')-\mathsf{L}(s',s)}. \qquad (48)$$

Hence a single double-flip, e.g. in $s \to s'$, can be balanced by two subsequent single-flips in $s' \to s''$ and $s'' \to s$ transitions. Since a phase may occur only in the double flip transition $s \to s'$, this particular contribution may be negative and would give $\langle \text{sign} \rangle < 1$.

Consequently, the single flip transitions provide an undesired background which indirectly causes negative signs of Boltzmann factors, hence the sign problem.

However such transitions vanish for $\epsilon \to 0$ having a weight of the higher order in ϵ by construction. This is clearly confirmed by our calculations, cf. Figs 11 and 12, and explains why sign problem vanishes at small ϵ. These figures were obtained by using Mathematica [64].

Moreover, by increasing the penalty parameter p we can force the "bad transitions" to vanish faster. Indeed this is also confirmed by our results for $p = 8$ in both dimensions. This suggests that the sign problem could be significantly reduced by setting $p = \infty$, which amounts to introducing a constraint in the Euclidean system [65].

We remark that even under this constraint there exist "Euclidean histories" with negative sign. As they involve a number of spin flips growing with the system size, one may hope that they do not lead to significant difficulties.

Obviously all these scenarios should be further studied quantitatively.

VI. CONCLUSIONS AND OUTLOOK

We have presented a bosonization method generalizing the idea from [4], valid for lattices of arbitrary coordination number and with arbitrary number of Majorana modes per lattice site. In the previous works only systems with even coordination numbers and one pair of fermionic creation/annihilation operators per lattice site were considered. The new approach extends the construction in several ways. First, for lattices with vertices of even degree we may include multiple fermionic states per site. We illustrate this by bosonizing the Hubbard model. Second, we allow for lattices with odd coordination numbers. Then there is an odd number of Majorana fermions per site. We stress that the Majorana variables we are talking about here are not necessary resulting from any representation of complex (Dirac) fermions, but they are the elementary objects per se. In particular systems with one Majorana per site may be bosonized. Since the presented bosonization procedure is clearly invertible (as it is based on an algebraic isomorphism), this leads to an intriguing possibility of analyzing other spin liquids by applying the inverse of it. We have illustrated this general phenomena on the simplest example, the Kitaev's honeycomb lattice, but one can apply this procedure to other models of this type. Similar constructions based on Clifford algebras formalism have been previously, as discussed in Sec.I, considered in [52–54] in order to fermionize higher spin models. More recently the gamma-matrix versions of Kitaev's models were used to study spin-$\frac{3}{2}$ Kitaev Shastry-Sutherland model [66] as well as to describe spin-orbital models and they relations to Kugel-Khomskii-type models and compass interactions [67]. In the latter case the authors constructed models, on either rectangular or honeycomb lattices, realizing the Kitaev's sixteenfold way of anyons [34]. Our bosonization method provides a rigorous mathematical technique that, in principle, could be use to generalize such construction in other geometries. Three-dimensional Kitaev's spin liquids were also studied recently in [68]. Several examples of possible use of Γ-Kitaev models to study higher spin models as well as spin-orbital models were also reported in [69], and used in [54, 70] to demonstrate the existence of emergent topological insulators on a three-dimensional diamond lattice. Since our bosonization provides tools for a rigorous construction (out of almost arbitrary fermionic theories) of bosonic (higher spin) models in terms of gamma matrices, it can be also used to generate new examples of (higher) spin models. We postpone this intriguing possibility for a future research.

It is possible to treat also systems for which the coordination number is not congruent modulo two to the number of Majoranas per site. Strictly speaking in this case we do not bosonize the original fermionic system but rather one augumented by some spurious fermionic degrees of freedom. Nevertheless, operators corresponding on the bosonic side to these modes may be clearly identified and decoupled. Even in the case of very regular lattices such trick is needed in presence of a boundary. We emphasize that this is a feature of our bosonization method, not of fermionic systems per se.

If these two numbers are not congruent modulo two, it involves spurious fermionic states, which nevertheless can be identified and eliminated. Another potential source of interest in this construction is that it provides new analytically tractable examples of spin systems featuring edge modes.

One question which remains unanswered is whether our construction may be dualized to some higher gauge theory. For systems with one fermion (and hence two Majoranas) per lattice site such picture of bosonization has been obtained in [2, 3].

Concerning the Euclidean formulation, our main conclusion is that in spite of somewhat unusual time evolution, generated by simultaneous double-flips, a local Euclidean action for an unconstrained system was derived. It contains at least six-spin interactions and is highly asymmetric between space and time, in contrast to the standard Ising model. To our knowledge, this system has not been studied. Now, it can be readily explored with standard statistical methods.

Our generic action is complex. It was found that the resulting sign problem is manageable on small lattices where our fully analytical approach is available.

The next logical step now is to study the problem for larger, although intermediate, sizes and see whether the popular reweighting methods allow meaningful measurements of observables, extrapolation to larger volumes and extraction of scaling limits. Moreover, it is conceivable that introduced here methods could be extended to implement the spin constraints avoiding the standard non-positive Legendre transformation. We intend to further study some of these questions with the aid of quantitative Monte Carlo approach.

ACKNOWLEDGMENTS

The authors acknowledge W. Brzezicki for suggesting the decagonal lattice as a three-dimensional example with odd degree vertices. AB acknowledges G. Ortiz and Z. Nussinov for pointing out the references [52–54] and for discussion about them. This work is supported in part by the NCN grant: UMO-2016/21/B/ST2/01492. BR was also supported by the MNS donation for PhD students and young scientists N17/MNS/000040.

[1] P. Jordan and E. Wigner, Über das paulische äquivalenzverbot, Z. Phys. **47**, 631 (1928).

[2] Y.-A. Chen, A. Kapustin, and D. Radičević, Exact bosonization in two spatial dimensions and a new class of lattice gauge theories, Annals of Physics **393**, 234 (2018).

[3] Y.-A. Chen and A. Kapustin, Bosonization in three spatial dimensions and a 2-form gauge theory, Phys. Rev. B **100**, 245127 (2019).

[4] J. Wosiek, A local representation for fermions on a lattice, Acta Phys. Polon. B **13**, 543 (1982).

[5] A. Bochniak and B. Ruba, Bosonization based on Clifford algebras and its gauge theoretic interpretation, JHEP **12**, 118, arXiv:2003.06905 [math-ph].

[6] C. Burgess, C. Lütken, and F. Quevedo, Bosonization in higher dimensions, Physics Letters B **336**, 18 (1994).

[7] S. B. Bravyi and A. Y. Kitaev, Fermionic quantum computation, Annals of Physics **298**, 210 (2002).

[8] R. C. Ball, Fermions without fermion fields, Phys. Rev. Lett. **95**, 176407 (2005).

[9] F. Verstraete and J. I. Cirac, Mapping local hamiltonians of fermions to local hamiltonians of spins, Journal of Statistical Mechanics: Theory and Experiment **2005**, P09012 (2005).

[10] E. Fradkin, Jordan-wigner transformation for quantum-spin systems in two dimensions and fractional statistics, Phys. Rev. Lett. **63**, 322 (1989).

[11] A. Karch and D. Tong, Particle-vortex duality from 3d bosonization, Phys. Rev. X **6**, 031043 (2016).

[12] E. Zohar and J. I. Cirac, Eliminating fermionic matter fields in lattice gauge theories, Phys. Rev. B **98**, 075119 (2018).

[13] Y.-A. Chen, Exact bosonization in arbitrary dimensions, Phys. Rev. Research **2**, 033527 (2020).

[14] D. T. Son, Is the composite fermion a dirac particle?, Phys. Rev. X **5**, 031027 (2015).

[15] J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu, Exact boson-fermion duality on a 3d euclidean lattice, Phys. Rev. Lett. **120**, 016602 (2018).

[16] D. C. Mattis and E. H. Lieb, Exact solution of a many-fermion system and its associated boson field, Journal of Mathematical Physics **6**, 304 (1965), https://doi.org/10.1063/1.1704281.

[17] E. Witten, Non-abelian bosonization in two dimensions, Commun. Math. Phys. **92**, 455 (1984).
[18] E. Cobanera, G. Ortiz, and Z. Nussinov, The bond-algebraic approach to dualities, Advances in Physics **60**, 679 (2011), https://doi.org/10.1080/00018732.2011.619814.
[19] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. part i, Phys. Rev. **60**, 252 (1941).
[20] W. Franz, Duality in generalized ising models and phase transitions without local order parameter, Journal of Mathematical Physics **12** (1971).
[21] A. Karch, D. Tong, and C. Turner, A Web of 2d Dualities: $\mathbf{Z}_2$ Gauge Fields and Arf Invariants, SciPost Phys. **7**, 7 (2019).
[22] N. Seiberg, T. Senthil, C. Wang, and E. Witten, A duality web in 2+1 dimensions and condensed matter physics, Annals of Physics **374**, 395 (2016).
[23] T. Senthil, D. T. Son, C. Wang, and C. Xu, Duality between $(2+1)d$ quantum critical points, Physics Reports **827**, 1 (2019), duality between $(2+1)d$ quantum critical points.
[24] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Two-dimensional ising model as a soluble problem of many fermions, Rev. Mod. Phys. **36**, 856 (1964).
[25] Y. Delgado, C. Gattringer, and A. Schmidt, Solving the sign problem of two flavor scalar electrodynamics at finite chemical potential, PoS **LATTICE 2013**, 147 (2014).
[26] C. Gattringer, T. Kloiber, and V. Sazonov, Solving the sign problems of the massless lattice schwinger model with a dual formulation, Nuclear Physics B **897**, 732 (2015).
[27] A. Kitaev, Fault-tolerant quantum computation by anyons, Annals of Physics **303**, 2 (2003).
[28] A. Kitaev and C. Laumann, Topological phases and quantum computation (2009), arXiv:0904.2771 [cond-mat.mes-hall].
[29] D. Gaiotto and A. Kapustin, Spin TQFTs and fermionic phases of matter, Int. J. Mod. Phys. A **31**, 1645044 (2016), arXiv:1505.05856 [cond-mat.str-el].
[30] T. Mazaheri, G. Ortiz, Z. Nussinov, and A. Seidel, Zero modes, bosonization, and topological quantum order: The laughlin state in second quantization, Phys. Rev. B **91**, 085115 (2015).
[31] A. Kapustin and R. Thorngren, Fermionic SPT phases in higher dimensions and bosonization, JHEP **10**, 080, arXiv:1701.08264 [cond-mat.str-el].
[32] A. Cappelli, E. Randellini, and J. Sisti, Three-dimensional topological insulators and bosonization, J. High Energ. Phys. **135**.
[33] O. Pozo, P. Rao, C. Chen, and I. Sodemann, Anatomy of $\mathbb{Z}_2$ fluxes in anyon fermi liquids and bose condensates, Phys. Rev. B **103**, 035145 (2021).
[34] A. Kitaev, Anyons in an exactly solved model and beyond, Annals Phys. **321**, 2 (2006), arXiv:cond-mat/0506438.
[35] S. Huber and M. Kollar, From luttinger liquids to luttinger droplets via higher-order bosonization identities, Phys. Rev. Research **2**, 043336 (2020).
[36] T. Bortolin, A. Iucci, and A. M. Lobos, Quantum phase diagram of shiba impurities from bosonization, Phys. Rev. B **100**, 155111 (2019).
[37] B. Yang, Statistical interactions and boson-anyon duality in fractional quantum hall fluids, Phys. Rev. Lett. **127**, 126406 (2021).
[38] A. M. Szczerba, Spins and fermions on arbitrary lattices, Commun. Math. Phys. **98**, 513 (1985).
[39] A. Bochniak, B. Ruba, J. Wosiek, and A. Wyrzykowski, Constraints of kinematic bosonization in two and higher dimensions, Phys. Rev. D **102**, 114502 (2020), arXiv:2004.00988 [hep-lat].
[40] E. Majorana, Teoria simmetrica dell'elettrone e del positrone, Nuovo Cim **14**, 171 (1937).
[41] More precisely, complex structure compatible with the bilinear form determining the canonical anticommutation relations.
[42] J. R. Schrieffer, *Theory of Superconductivity* (W. A. Benjamin, 1964).
[43] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Theory of superconductivity, Phys. Rev. **108**, 1175 (1957).
[44] A. A. Abrikosov, On the Magnetic properties of superconductors of the second group, Sov. Phys. JETP **5**, 1174 (1957).
[45] F. Wilczek, Majorana returns, Nature Phys , 614–618 (2009).
[46] N. Read and D. Green, Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect, Phys. Rev. B **61**, 10267 (2000).
[47] N. B. Kopnin and M. M. Salomaa, Mutual friction in superfluid ^{3}He: Effects of bound states in the vortex core, Phys. Rev. B **44**, 9667 (1991).
[48] G. Moore and N. Read, Nonabelions in the fractional quantum hall effect, Nuclear Physics B **360**, 362 (1991).
[49] A. Y. Kitaev, Unpaired majorana fermions in quantum wires, Physics-Uspekhi **44**, 131 (2001).
[50] S. Sarma, M. Freedman, and C. Nayak, Majorana zero modes and topological quantum computation, npj Quantum Inf **1**, 15001 (2015).
[51] H.-L. Huang, M. Narożniak, F. Liang, Y. Zhao, A. D. Castellano, M. Gong, Y. Wu, S. Wang, J. Lin, Y. Xu, H. Deng, H. Rong, J. P. Dowling, C.-Z. Peng, T. Byrnes, X. Zhu, and J.-W. Pan, Emulating quantum teleportation of a majorana zero mode qubit, Phys. Rev. Lett. **126**, 090502 (2021).
[52] Z. Nussinov and G. Ortiz, Bond algebras and exact solvability of hamiltonians: Spin $s = \frac{1}{2}$ multilayer systems, Phys. Rev. B **79**, 214440 (2009).
[53] H. Yao, S.-C. Zhang, and S. A. Kivelson, Algebraic spin liquid in an exactly solvable spin model, Phys. Rev. Lett. **102**, 217202 (2009).
[54] C. Wu, D. Arovas, and H.-H. Hung, Γ-matrix generalization of the kitaev model, Phys. Rev. B **79**, 134427 (2009).
[55] F. Wilczek, Magnetic flux, angular momentum, and statistics, Phys. Rev. Lett. **48**, 1144 (1982).
[56] J. Hubbard, Electron correlations in narrow energy bands, Proc. R. Soc. Lond. A **276**, 238–257 (1963).
[57] H. Nielsen and M. Ninomiya, Absence of neutrinos on a lattice: (i). proof by homotopy theory, Nuclear Physics B **185**, 20 (1981).
[58] D. Friedan, A proof of the nielsen-ninomiya theorem, Communications in Mathematical Physics **85**, 481 (1982).
[59] A. Y. Kitaev, Unpaired majorana fermions in quantum wires, Physics-Uspekhi **44**, 131 (2001).
[60] E. Fradkin and L. Susskind, Order and disorder in gauge systems and magnets, Phys. Rev. D **17**, 2637 (1978).
[61] J. B. Kogut, An introduction to lattice gauge theory and

spin systems, Rev. Mod. Phys. **51**, 659 (1979).

[62] D. P. Landau and K. Binder, *A Guide to Monte Carlo Simulations in Statistical Physics* (Cambridge University Press, 2014).

[63] T. Nakamura, N. Hatano, and H. Nishimori, Reweighting method for quantum monte carlo simulations with the negative-sign problem, Journal of the Physical Society of Japan **61**, 3494 (1992).

[64] W. R. Inc., System Modeler, Version 12.1, champaign, IL, 2020.

[65] Not to be confused with plaquette constraints required for bosonization.

[66] T. Eschmann, V. Dwivedi, H. F. Legg, C. Hickey, and S. Trebst, Partial flux ordering and thermal majorana metals in higher-order spin liquids, Phys. Rev. Research **2**, 043159 (2020).

[67] S. Chulliparambil, U. F. P. Seifert, M. Vojta, L. Janssen, and H.-H. Tu, Microscopic models for kitaev's sixteenfold way of anyon theories, Phys. Rev. B **102**, 201111 (2020).

[68] T. Eschmann, P. A. Mishchenko, K. O'Brien, T. A. Bojesen, Y. Kato, M. Hermanns, Y. Motome, and S. Trebst, Thermodynamic classification of three-dimensional kitaev spin liquids, Phys. Rev. B **102**, 075125 (2020).

[69] G.-W. Chern, Three-dimensional topological phases in a layered honeycomb spin-orbital model, Phys. Rev. B **81**, 125134 (2010).

[70] S. Ryu, Three-dimensional topological phase on the diamond lattice, Phys. Rev. B **79**, 075124 (2009).