

Jagiellonian University in Krakow
Faculty of Physics, Astronomy and Applied Computer Science

WOJCIECH TARNOWSKI
Student number: 1079390

DIFFUSION ON MATRICES IN THE GINIBRE ENSEMBLE

Bachelor thesis
field of study: Theoretical Physics

Thesis written under the supervision of
prof. dr hab. Maciej A. Nowak
Mark Kac Complex Systems Research Centre

Krakow, June 2014

Oświadczenie autora pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Kraków, dnia

Podpis autora pracy

Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Kraków, dnia

Podpis kierującego pracą

Contents

| | | |
|----------|---|-----------|
| 1 | Preface | 4 |
| 2 | Introduction | 5 |
| 2.1 | Non-Hermitian random matrices | 5 |
| 2.2 | Ginibre diffusion | 6 |
| 3 | Analytic structure | 7 |
| 3.1 | Averaged determinant | 7 |
| 3.2 | Diffusion equation for D | 7 |
| 3.3 | Further transformation | 8 |
| 3.4 | Solutions | 9 |
| 4 | Applications | 10 |
| 4.1 | Girko-Ginibre | 10 |
| 4.2 | Deterministic matrix | 13 |
| 4.3 | Band matrix | 14 |
| 5 | Summary | 16 |
| 6 | Acknowledgments | 16 |
| | Bibliography | 17 |
| | Appendices | 18 |
| A | Grassmann algebra | 18 |

1 Preface

A random walk on matrices has been considered recently, miscellaneous models including additive and multiplicative Brownian motion has been studied on various classes of symmetry [1, 2]. Astonishing results unveiled an elaborate evolution of the spectral density, including nonlinear partial differential equation of motion and shock waves among solutions [3, 4, 5], despite the classical linear diffusion of matrix elements.

This paper takes under consideration the most general case in complex matrices of no imposed symmetry constrains and no restrictions to real and imaginary parts of elements. We present an analytical approach making use of a complex variable that hitherto was treated as a regulator only and put to zero after taking the limit of infinite size of matrices. This underappreciated variable encodes a nonlinear dynamics of the correlator and an appearance of the shock corresponds to a non-vanishing eigenvalue distribution. The work presented in this thesis comprises a part of a paper submitted to Physical Review Letters [6].

In the first part, an introduction, we give a reader a brief insight into non-Hermitian random matrix ensembles, define the process of the Ginibre diffusion and introduce main objects describing spectral properties. In the next section, we derive the announced formalism of the random walk in the natural objects - the eigenvalue density and the correlator. We reduce the problem to the one-dimensional Burgers' equation and present the general way of solving. In the last part we apply this approach to study the evolution with different initial conditions. Obtained results are compared to Monte Carlo simulations.

In order to avoid the disambiguity, the variable referring to a matrix is denoted by a double stroke font.

2 Introduction

2.1 Non-Hermitian random matrices

Matrices with no imposed symmetry constraints can have in general complex eigenvalues. Since their spectrum is two-dimensional, this is not uniquely determined by all its moments, therefore the Stieltjes transform does not apply in this problem. Despite the crucial disparity between Hermitian and non-Hermitian matrices, there exists a common construction that gives a facility to treat them in a similar manner.

Entries in the form of $X_{jk} = x_{jk} + iy_{jk}$ of the considered matrices of size N are generated from a certain probability density function

$$P(\mathbb{X})d\mathbb{X} = \int c_N e^{-\frac{1}{N}\text{Tr}V(\mathbb{X}, \mathbb{X}^\dagger)} \prod_{j,k=1}^N dx_{jk} \prod_{l,m=1}^N dy_{lm}. \quad (1)$$

Due to the cyclicity of a trace, the ensemble is invariant under the similarity transformations: $\mathbb{X} \rightarrow \mathbb{P}\mathbb{X}\mathbb{P}^{-1}$. With probability 1 one can find the matrix \mathbb{P} that brings \mathbb{X} to a diagonal form. Such an operation is equivalent to the change of variables in the integral. Because of the tracial invariance, the Jacobian factorizes and all irrelevant parameters of the transformation can be integrated out. Finally, the joint probability density function for eigenvalues takes the form of

$$P(\lambda_1, \dots, \lambda_N) = c'_N \int \exp \left(-\frac{1}{N} \sum_{j=1}^N V(\lambda_j, \bar{\lambda}_j) \right) \prod_{j \neq k} |\lambda_j - \lambda_k|^2 \prod_{j=1}^N d\lambda_j. \quad (2)$$

The Vandermonde, interpreted by Dyson [7] as a repulsive 2D electrostatic potential, is a link to Hermitian matrices pointing out that the electrostatic analogy can be extended to this class of matrices. An one-dimensional delta function needs to be replaced by its complex representation because of the richer structure of the eigenvalue density, defined by

$$\rho(z) = \frac{1}{N} \left\langle \sum_{i=1}^N \delta^{(2)}(z - \lambda_i) \right\rangle. \quad (3)$$

The averaging is taken with respect to the probability determining the ensemble

$$\langle f(x) \rangle = \int f(x) P(\mathbb{X}) d\mathbb{X}. \quad (4)$$

A construction exploiting the electrostatic analogy was introduced in [8] where the electrostatic potential was defined as:

$$\varphi(z, \bar{z}) = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \left\langle \frac{1}{N} \text{Tr} \log \left(|z\mathbb{1} - \mathbb{X}|^2 + \epsilon^2 \mathbb{1} \right) \right\rangle. \quad (5)$$

The spectral density can be recovered via a Poisson law: $\rho(\lambda) = \frac{1}{\pi} \partial_z \bar{\partial}_{\bar{z}} \varphi(z, \bar{z})$. Limits in the expression above need to be taken in a proper order, otherwise the whole information stored in the potential will be lost.

In the analogy to Hermitian matrices, one would define the electric field as a gradient of the potential $E = \partial_z \varphi$, however such an object is inconvenient in practical calculations because of the quadratic expression in the denominator. An alternative way simplifying this difficulty is the linearization through letting the resolvent be a 2×2 matrix:

$$\mathcal{G}(\mathbb{Q}) := \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix} = \frac{1}{N} \left\langle \text{bTr} \frac{1}{\mathbb{Q} \otimes \mathbb{1} - \mathbb{H}} \right\rangle := \frac{1}{N} \begin{pmatrix} \left\langle \text{Tr} \frac{\bar{z}\mathbb{1} - \mathbb{X}^\dagger}{\mathbb{E}} \right\rangle & \left\langle \text{Tr} \frac{\bar{w}\mathbb{1}}{\mathbb{E}} \right\rangle \\ \left\langle \text{Tr} \frac{-w\mathbb{1}}{\mathbb{E}} \right\rangle & \left\langle \text{Tr} \frac{z\mathbb{1} - \mathbb{X}}{\mathbb{E}} \right\rangle \end{pmatrix}. \quad (6)$$

Here \mathbb{Q} is the matrix representation of quaternion \otimes - the Kronecker product, \mathbb{H} a $2N \times 2N$ matrix and \mathbb{E} the expression in the logarithm in (5). Their explicit forms are as follows:

$$\mathbb{Q} = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}, \quad \mathbb{H} = \begin{pmatrix} \mathbb{X} & 0 \\ 0 & \mathbb{X}^\dagger \end{pmatrix}, \quad \mathbb{E} = (z\mathbb{1} - \mathbb{X})(\bar{z}\mathbb{1} - \mathbb{X}^\dagger) + |w|^2\mathbb{1}. \quad (7)$$

The upper-diagonal element of the resolvent yields the spectral density via the Gauss law $\rho(z) = \frac{1}{\pi} \partial_{\bar{z}} \mathcal{G}_{11}$. Since the matrix \mathbb{X} is not Hermitian it has distinct left ($\langle L|$) and right ($|R\rangle$) eigenvectors which form biorthogonal sets. In the infinite size limit a product of off-diagonal elements leads to a correlator of eigenvectors [12, 13]:

$$O(z) := \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_{j=1}^N \langle L_j | L_j \rangle \langle R_j | R_j \rangle \delta(z - \lambda_j) \right\rangle = \frac{N}{\pi} \mathcal{G}_{1\bar{1}} \mathcal{G}_{\bar{1}1}. \quad (8)$$

2.2 Ginibre diffusion

We consider here the diffusion process on matrices in a sense that both real and imaginary parts of each matrix element undergo a continuous random walk. Let \mathbb{X} be a non-Hermitian $N \times N$ matrix of elements $X_{jk} = x_{jk} + iy_{jk}$, their motion is driven by a noise:

$$dx_{jk} + idy_{jk} = \frac{1}{\sqrt{2N}} dB_{jk}^{(x)} + \frac{i}{\sqrt{2N}} dB_{jk}^{(y)}. \quad (9)$$

Due to the symmetry with respect to an interchange $x \leftrightarrow y$ any equation is written with the use of x variable, however one has to remember that exactly the same relation holds for y .

A probability for each element evolves independently according to a heat equation:

$$\partial_\tau P(x, \tau | x, 0) = \frac{1}{4N} \partial_{xx} P(x, \tau | x, 0). \quad (10)$$

The expected value of a displacement is zero, while the variance increases linearly with time. The average is taken with respect to the solution of (10):

$$\langle \delta x \rangle = 0 \quad \langle \delta x^2 \rangle = \frac{1}{2N} \tau. \quad (11)$$

The equation (10) can be easily solved in the case of a zero matrix as an initial condition: $P(x, 0) = \delta(x)$. The probability for each element is the Gaussian function:

$$P(x, \tau | \delta(x), 0) = \left(\frac{N}{\pi\tau} \right)^{1/2} \exp \left(-\frac{Nx^2}{\tau} \right). \quad (12)$$

The joint probability for the entire matrix is nothing but the product of pdfs for each element and thus given by:

$$P(\mathbb{X}, \tau) = \left(\frac{N}{\pi\tau} \right)^{N^2/2} \exp \left(-\frac{N}{\tau} \sum_{j,k=1}^N x_{jk}^2 + y_{jk}^2 \right) = \left(\frac{N}{\pi\tau} \right)^{N^2/2} \exp \left(-\frac{N}{\tau} \text{Tr} \mathbb{X} \mathbb{X}^\dagger \right). \quad (13)$$

The obtained result recovers the well-known Ginibre ensemble (GE), firstly introduced in [9]. This result shows that any problem of a matrix undergoing the diffusion described by the equation (9) can be reduced to an addition of a matrix \mathbb{X}_0 representing the initial conditions and the Ginibre matrix rescaled by a factor of $\sqrt{\tau}$.

The main areas of interests of random matrix theory are the spectral density and higher order correlation functions that can be derived by the integration of (2) which is a very non-trivial task, since in many cases the potential defining the probability density

function is unknown or appears as an expression that cannot be calculated explicitly. Free Probability Calculus, introduced by Voiculescu [11] and extended to non-Hermitian matrices [8], provides a powerful tool for dealing with that problem. If the spectral density or rather the Green's function of the initial matrix is known, the derivation of the spectral density of a diffusing matrix reduces to a free convolution. An example of that approach to the Brownian motion with a non-trivial boundary condition can be found in [14].

Having known that the procedure for solving this problem can be simply algorithmized, and any possible obstacles stem only from the fact that solutions of certain class of algebraic equations cannot be obtained in an explicit form, a natural seems to be the doubt if another approach brings anything new. In this paper we present a novel description of the diffusion, exploiting an analytic structure in the variable that everyone used to put zero.

3 Analytic structure

3.1 Averaged determinant

Let us define the following object which is a quaternion generalization of a characteristic polynomial:

$$D(z, w, \tau) = \langle \det(\mathbb{Q} - \mathbb{H}) \rangle_\tau = \left\langle \det(|z\mathbb{1} - \mathbb{X}|^2 + |w|^2\mathbb{1}) \right\rangle_\tau. \quad (14)$$

Variables z and w are two independent complex ones and \mathbb{X} is the diffusing matrix. The averaging operation $\langle \dots \rangle_\tau$ denotes the average with respect to the probability density function of the matrix \mathbb{X} after time τ . One has to notice that this is not the average over the time interval. In other words, the pdf of the initial matrix uniquely determines the pdf for any element $P(x, 0)$. In the next step one solves the equation (10) with the initial condition $P(x, 0)$. The solution is just a convolution of the initial pdf with a normal distribution (Green's function for the heat equation) and determines pdf in time τ . Such functions for each elements compose the joint pdf of the matrix after time τ :

$$P(\mathbb{X}, \tau | \mathbb{X}, 0) = \prod_{j,k=1}^N P(X_{jk}, \tau | X_{jk}, 0). \quad (15)$$

Hereafter, for the simplicity of the notation, the argument representing the initial condition is omitted and the flat measure of real and imaginary parts of matrix elements is denoted as:

$$\mathcal{D}[\mathbb{X}] = \prod_{j,k=1}^N dx_{jk} dy_{jk}. \quad (16)$$

We remark that the determinant does not depend on the w variable itself, but on its modulus, nevertheless the whole derivation of the equations is conducted in the general case.

3.2 Diffusion equation for D

Making use of the Grassmann variables (for a brief review see Appendix A), in particular the integral representation of the determinant, the object D can be expressed as follows:

$$D = \int P(\mathbb{X}, \tau) \exp \left[(\eta^T \ \xi^T) \begin{pmatrix} z\mathbb{1} - \mathbb{X} & -\bar{w}\mathbb{1} \\ w\mathbb{1} & \bar{z}\mathbb{1} - \mathbb{X}^\dagger \end{pmatrix} \begin{pmatrix} \zeta \\ \nu \end{pmatrix} \right] \mathcal{D}[\mathbb{X}] \mathcal{D}[\eta, \xi, \zeta, \nu] \quad (17)$$

Since the joint pdf is a product of probabilities for each element separately, a differentia-

tion with respect to τ obeys the Leibniz rule hence we have a sequence of equalities:

$$\begin{aligned}\partial_\tau P(\mathbb{X}, \tau) &= \sum_{i,j=1}^N \prod_{\substack{k \neq i \\ l \neq j}} P(X_{kl}, \tau) \partial_\tau P(X_{ij}, \tau) = \\ \sum_{i,j} \prod_{\substack{k \neq i \\ l \neq j}} P(X_{kl}, \tau) \frac{1}{4N} (\partial_{x_{ij}x_{ij}} + \partial_{y_{ij}y_{ij}}) P(x_{ij} + iy_{ij}, \tau) &= \frac{1}{4N} \mathcal{A}P(\mathbb{X}, \tau),\end{aligned}\quad (18)$$

where the operator \mathcal{A} is the $2N^2$ -dimensional Laplacian:

$$\mathcal{A} = \sum_{i,j=1}^N \partial_{x_{ij}x_{ij}} + \partial_{y_{ij}y_{ij}} \quad (19)$$

Let us calculate the argument of the exponent in (17) explicitly and denote it by R . After some algebra, one gets:

$$R = \sum_{i=1}^N z\eta_i\zeta_i - \bar{w}\eta_i\nu_i + w\xi_i\zeta_i + \bar{z}\xi_i\nu_i + \sum_{j,k=1}^N iy_{jk}(\nu_j\xi_k - \eta_j\zeta_k) - x_{jk}(\eta_j\zeta_k + \nu_j\xi_k). \quad (20)$$

We differentiate the object D with respect to τ and make use of (18). For the sake of simplicity, the measure is denoted by $\mathcal{D}[\dots]$.

$$\partial_\tau D = \int \partial_\tau P(\mathbb{X}, \tau) e^R \mathcal{D}[\dots] = \int \frac{1}{4N} e^R \mathcal{A}P(\mathbb{X}, \tau) \mathcal{D}[\dots] = \int \frac{1}{4N} P(\mathbb{X}, \tau) \mathcal{A}e^R \mathcal{D}[\dots], \quad (21)$$

where in the last step we integrated by parts twice and exploited the fact that P as a probability density function has to vanish at infinity. Indeed, for any pair of indices i, j we have

$$\int e^R \partial_{x_{ij}x_{ij}} P(\mathbb{X}, \tau) \mathcal{D}[\dots] = - \int \partial_{x_{ij}} e^R \partial_{x_{ij}} P(\mathbb{X}, \tau) \mathcal{D}[\dots] = \int P(\mathbb{X}, \tau) \partial_{x_{ij}x_{ij}} e^R \mathcal{D}[\dots]. \quad (22)$$

We study how the differential operators act on the exponent R :

$$\begin{aligned}\partial_{x_{jk}x_{jk}} R &= (\eta_j\zeta_k + \nu_j\xi_k)(\eta_j\zeta_k + \nu_j\xi_k) = 2\eta_j\zeta_k\nu_j\xi_k \\ \partial_{x_{jk}x_{jk}} R &= i^2(\nu_j\xi_k - \eta_j\zeta_k)(\nu_j\xi_k - \eta_j\zeta_k) = 2\eta_j\zeta_k\nu_j\xi_k.\end{aligned}\quad (23)$$

On the other hand, up to the factor of 4, the same quantity is attainable by the differentiation over w and \bar{w} :

$$\partial_{w\bar{w}} e^R = e^R \sum_{j,k} -\xi_k\zeta_k\eta_j\nu_j = e^R \sum_{j,k} \eta_j\zeta_k\nu_j\xi_k \quad (24)$$

Comparing (24) with (23) and substituting into (21) one finally is led to the equation:

$$\partial_\tau D = \frac{1}{N} \partial_{w\bar{w}} D \quad (25)$$

3.3 Further transformation

In the $N \rightarrow \infty$ limit one can first calculate the average and then take a logarithm of D and get the same result when making these two operations in a reverse order. Therefore the potential (5) can be also obtained from D .

$$\varphi(z) = \lim_{w \rightarrow 0} \lim_{N \rightarrow \infty} \log D(z, w). \quad (26)$$

In order to take advantage of this correspondence we introduce an auxiliary variable $u = \frac{1}{N} \log D$ which transforms (25) into

$$\partial_\tau u = \partial_w u \partial_{\bar{w}} u + \frac{1}{N} \partial_{w\bar{w}} u. \quad (27)$$

Despite that a linear diffusion equation (25) was mapped into a nonlinear (noiseless Kardar-Parisi-Zhang) one, u allows one to calculate the spectral density without calculating complicated integrals such as (2). Let us also remark that the matrix $|z\mathbb{1} - \mathbb{X}|^2 + |w|^2\mathbb{1}$ is positive definite, therefore D takes values from the interval $(0, +\infty)$ and, as a consequence, u is real-valued. We introduce another two auxiliary variables

$$v = \partial_w u \quad (28)$$

$$g = \partial_z u, \quad (29)$$

which satisfy the following equations:

$$\partial_\tau v = \frac{1}{N} \partial_{w\bar{w}} v + \partial_w |v|^2 \quad (30)$$

$$\partial_\tau g = \frac{1}{N} \partial_{w\bar{w}} g + \partial_z |v|^2. \quad (31)$$

These quantities are counterparts of the quaternion resolvent (6) corresponding to matrices of a finite size. As it was already remarked, u recreates the potential in $N \rightarrow \infty$ limit. Moreover, after putting $w \rightarrow 0$, g tends to \mathcal{G}_{11} , whereas v to $\mathcal{G}_{1\bar{1}}$. Even though governing equations became more complicated, in particular nonlinear, the presented objects are natural variables in a description of spectral properties of random matrices. Although any equation written so far is exact for finite N , the spectral density and the correlator can be obtained solely in $N \rightarrow \infty$ limit.

In order to find the spectral density and the correlator, one needs first to solve this system of equations for finite N and then take $N \rightarrow \infty$ limit. The theory of partial differential equations states that such a procedure leads to the function which is almost everywhere a solution of this system when the limit is taken in the level of the equation and the initial condition. For the infinite size of matrices, the equations simplify considerably:

$$\partial_\tau u = \partial_w u \partial_{\bar{w}} u \quad (32)$$

$$\partial_\tau v = \partial_w |v|^2 \quad (33)$$

$$\partial_\tau g = \partial_z |v|^2. \quad (34)$$

The last two equations are not independent, since by construction (29) variables v and g fulfill the relation

$$\partial_z v = \partial_w g. \quad (35)$$

3.4 Solutions

Variables v and g are complex-valued, however their algebraic structure is not exploited on the level of the differential equation, thus they can be regarded as pairs of real-valued functions. Obtained equations involve two spatial dimensions and one temporal, their form, especially (33), resembles an one-dimensional Burgers' equation - a simple model proposed to describe shockwaves and turbulence. With the use of this picture, the term $\frac{1}{N}$ at the laplacian in (27) (31) (31) can be considered as a viscosity.

We remind the observation that the averaged determinant depends on the modulus $|w|$ thus it is convenient to represent it in polar coordinates $w = re^{i\theta_w}$. Due to that symmetry, the equation (32) reduces to:

$$\partial_\tau u = \frac{1}{4} (\partial_r u)^2. \quad (36)$$

The solution can be easily found from the initial condition $u(z, r, \tau = 0)$ by the Hopf-Lax formula [15, 16]:

$$u(z, r, \tau) = \sup_{\xi \geq 0} \left(u(z, \xi, 0) - \frac{(r - \xi)^2}{\tau} \right). \quad (37)$$

The time derivative in the equation for v (33) depends on its modulus solely and the initial condition is independent of the polar variable. We decompose v on its radial and angular parts $v = |v|e^{i\theta_v}$, direct calculations show that because of the cylindrical symmetry in the boundary condition, there is no evolution in θ , only $|v|$ varies in time, according to the equation:

$$\partial_\tau |v| = |v| \partial_r |v|. \quad (38)$$

The problem was reduced to the one dimensional Burgers' equation, which can be solved by the method of characteristics. The domain of the equation was brought to the positive semi axis in r , however the rotational symmetry imposes that modulus of v at the opposite (with respect to the origin) points is the same. This remark allows to extend the problem to the full space of r , which eliminates problematic boundary conditions at $r = 0$. Not only the initial conditions are known on the whole $r = 0$ line but also they are symmetric. Obviously, the time is confined to the $\tau \geq 0$ half-axis. Curves in the (r, τ) plane, along which the solution is constant, belong to the one parameter family

$$r + \tau |v|(r, \tau) = \text{const}. \quad (39)$$

These are straight lines of a different slope and y -intercepts. Substituting $\tau \rightarrow 0$ above we get the value of the constant as an intersection point with the r -axis, let us denote this ξ . Since the solution is constant along such a line, one is finally led to the equation for the family:

$$F(r, \tau; \xi) = -r + \tau |v_0(\xi)| - \xi = 0. \quad (40)$$

Depending on the initial condition $|v_0|$, characteristics can cross, leading to the multivaluedness of the solution or the appearance of points unattainable from the boundary. Such a behavior can be avoided by constructing shocks - lines at which the solution is discontinuous and characteristics are terminated. The region where this phenomenon occurs is bounded by an envelope, also known as pre-shock or caustic. These curves can be found by the condition

$$\frac{\partial F}{\partial \xi} = 0. \quad (41)$$

The solution fulfilling the initial condition $|v|(r, 0) = |v_0|(r)$ is the solution of the algebraic equation:

$$|v| = |v_0(r + \tau |v|)|. \quad (42)$$

4 Applications

4.1 Girko-Ginibre

We first consider trivial initial conditions with a zero matrix. As it was already discussed in the introduction, this case recovers Girko-Ginibre ensemble. The initial values of the averaged determinant and other auxiliary variables read:

$$D_0(z, w) = \left(|z|^2 + |w|^2\right)^N, \quad u_0(z, w) = \log \left(|z|^2 + |w|^2\right), \quad v_0(z, w) = \frac{\bar{w}}{|z|^2 + |w|^2}. \quad (43)$$

Characteristics and caustics of the Burgers' equation (38) given by these initial conditions are depicted in the figure 1. The picture shows only the intersection of the whole w plane corresponding to two polar angles θ_w and $\pi + \theta_w$. The position of the emerging shock is determined by Rankine-Hugoniot conditions. Due to the rotational symmetry, it is located at $|w| = 0$ and form at critical time $\tau_c = |z|^2$. During the whole evolution z variable representing position on the complex plane where the spectral density is considered, here serves as a parameter only.

One can see from the picture that there is a vertical characteristic starting from $r = 0$ of value $|v| = 0$ and at the critical time reaches the shock and becomes truncated. The solution at this point is determined by another initial condition.

The function $|v|$ can be obtained from the cubic equation (42):

$$|v| = \frac{r + \tau|v|}{|z|^2 + (r + \tau|v|)^2}. \quad (44)$$

Its real roots are depicted in the fig. 2. Solutions are valid in the $r \geq 0$ region. After the critical time a disambiguity arises and a domain of the solution is broader than the positive semi axis. The existence of the shock retains characteristics to cross the point $r = 0$ and the additional solution is invalid.

Substituting $r = 0$ in the eq. (44) one obtains the off-diagonal elements in Green's function (6). The zero solution is valid until the shock appears, then the non-zero one is proper.

$$|v|(z, r = 0) = \begin{cases} \frac{1}{\tau} \sqrt{\tau - |z|^2} & \text{for } |z|^2 \leq \tau \\ 0 & \text{for } |z|^2 > \tau \end{cases} \quad (45)$$

This quantity allows one to derive the eigenvector correlator (8):

$$O(z) = \frac{N}{\pi} |v|^2 = \begin{cases} \frac{N}{\pi \tau^2} (\tau - |z|^2) & \text{for } |z|^2 \leq \tau \\ 0 & \text{for } |z|^2 > \tau \end{cases} \quad (46)$$

One can substitute the solution of (44) to the eq. (34) and solve it, however the Cardano's formula does not yield a concise and treatable form. We propose another method of solving, relying on the Hopf-Lax formula (37), which reduces the problem to the finding of the supremum. Since only the value in $r = 0$ has an interpretation, we put it before determining the maximum, which simplifies calculations significantly.

$$u(z, 0, \tau) = \sup_{\xi \geq 0} \left(\log(|z|^2 + \xi^2) - \frac{\xi^2}{\tau} \right). \quad (47)$$

We remind that z and τ are treated only as parameters, although the resultant value intimately depends on them:

$$u(z, 0, \tau) = \begin{cases} \log \tau - 1 + \frac{|z|^2}{\tau} & \text{for } |z|^2 \leq \tau \\ \log |z|^2 & \text{for } |z|^2 > \tau \end{cases} \quad (48)$$

Further differentiation with respect to z and \bar{z} gives the spectral density:

$$\rho(z) = \frac{1}{\pi \tau} \Theta(\tau - |z|^2). \quad (49)$$

The correlator can be derived through the same procedure, nevertheless one cannot put $r = 0$ before calculating the supremum so in this case the algebraic method is less time consuming. Both results were confirmed by numerical simulations which are shown in the fig. 3.

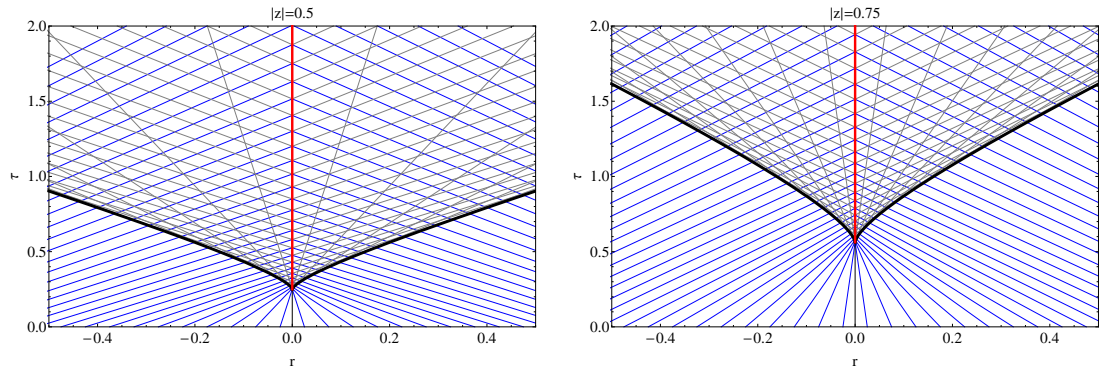


Figure 1: Characteristics of the Burgers' equation. Blue lines are the curves of a constant value of $|v|$, truncated on the shock (red line). Their extension is depicted as gray lines. The black curve, cusp, bounds the area where characteristics cross.

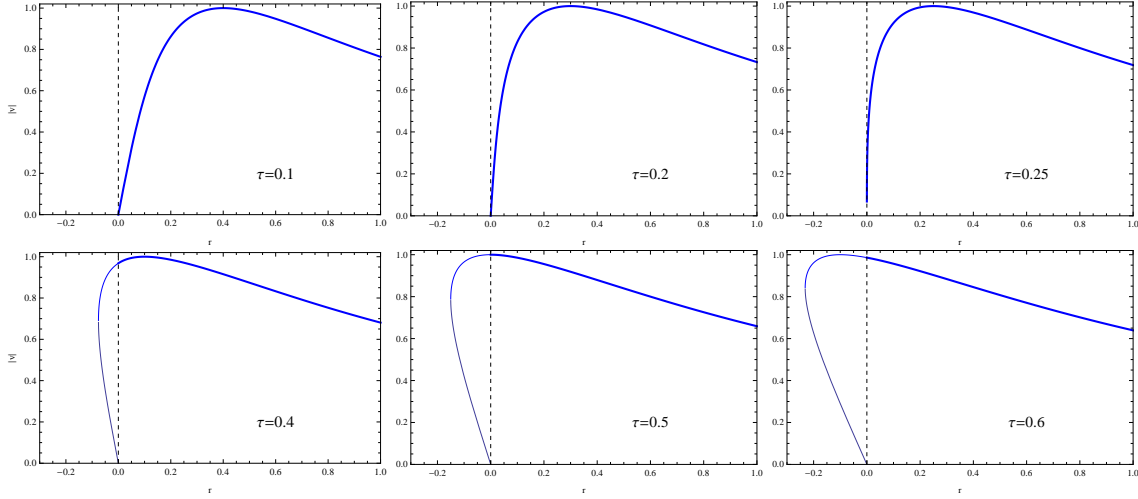


Figure 2: Solutions of the equation (44) for $|z| = 0.5$ at several different times. The thin line represents solution in the $r < 0$ region which is forbidden since r is the radial variable. This solution is truncated by the shock.

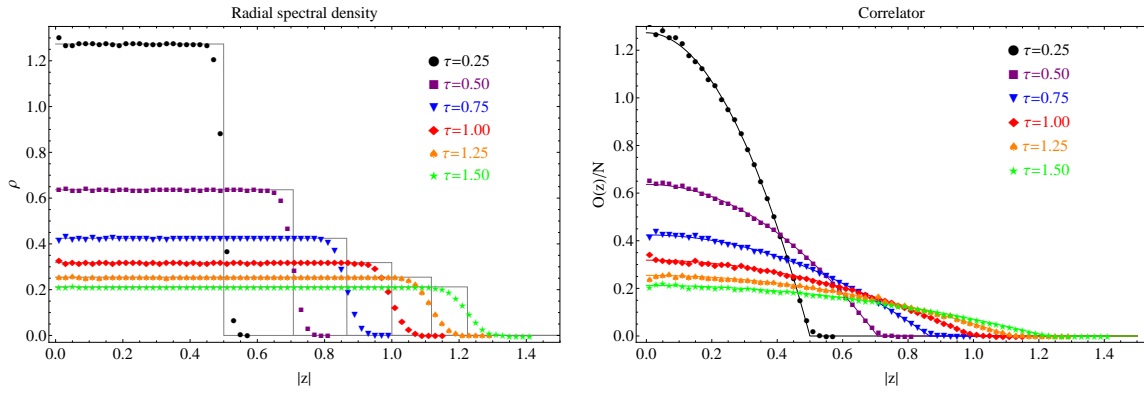


Figure 3: The eigenvalue density and the correlator in a radial variable. The solid line represents theoretical results, numerical data were obtained by diagonalization $3 \cdot 10^4$ matrices of size $N = 200$ for the parameter $a = 1$. The discrepancy near the edge of the support is an effect of finite N .

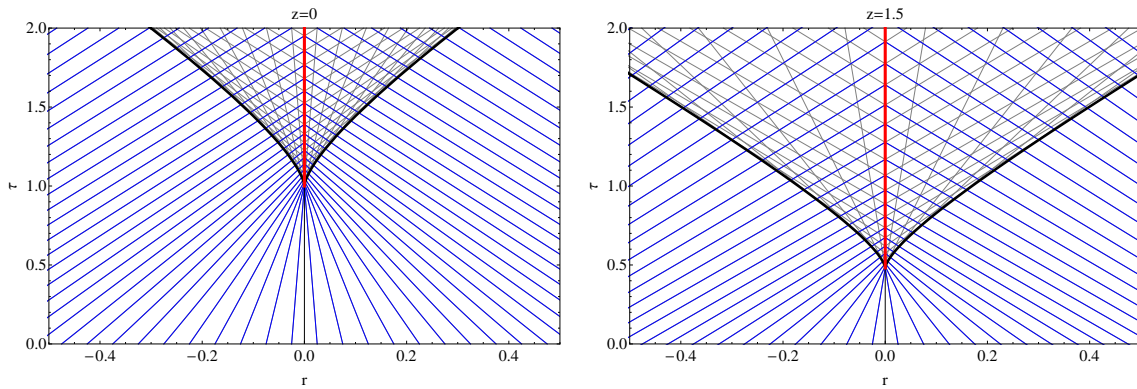


Figure 4: Characteristics in the problem of a diffusing deterministic matrix at two points on the z plane $z = 0$ and $z = 1.5$ for $a = 1$.

4.2 Deterministic matrix

We consider an initial matrix with two distinct eigenvalues $\pm a$, the number of both signs is equal, a can be in general complex. Such an example is interesting, since there is no rotational symmetry in z and the density support in the beginning consists of two disjoint sets that broaden with time and at a certain moment collide forming a connected set [14]. The initial conditions are as follows:

$$\begin{aligned} D_0 &= (|z - a|^2 + |w|^2)^{\frac{N}{2}} (|z + a|^2 + |w|^2)^{\frac{N}{2}}, \\ |v_0| &= \frac{\frac{1}{2}r}{|z - a|^2 + r^2} + \frac{\frac{1}{2}r}{|z + a|^2 + r^2}. \end{aligned} \quad (50)$$

Characteristics in this problem are depicted in the fig. 4. Due to the symmetry in w , the shock appears also at $r = 0$, although the initial conditions are so advanced that the position of the pre-shock cannot be calculated analitically. The shock emerges at a certain time, which can be obtained by putting $r = 0$ in the equation for the family of characteristics (40) juxtaposed with the envelope condition (41). One reads:

$$\tau = \frac{|a^2 - z^2|^2}{|a|^2 + |z|^2}. \quad (51)$$

This equation also determines the boundary of the eigenvalue support on a complex plane z at an arbitrary moment τ . The curve given by the equation above is called spiric section and is known as an intersection of a torus and a plane parallel to its rotational symmetry axis.

The function $|v|$ is a solution of the algebraic equation

$$2|v| = \frac{r + \tau|v|}{|z - a|^2 + (r + \tau|v|)^2} + \frac{r + \tau|v|}{|z + a|^2 + (r + \tau|v|)^2}, \quad (52)$$

which simplifies in the case $r = 0$. The zero solution is valid outside the area occupied by eigenvalues, there also exists a branch of positive solution, appropriate within the support, written below in its most accessible form:

$$|v|^2(r = 0) = \frac{1}{2} - \frac{1}{\tau^2} (|a|^2 + |z|^2) + \sqrt{\frac{1}{4} + \frac{1}{\tau^4} (a\bar{z} + z\bar{a})^2}. \quad (53)$$

In fact, this is, up to the constant factor $\frac{N}{\pi}$, the correlator. From the Hopf-Lax formula (37) for the KPZ equation we get

$$u(z, \tau) = \sup_{\xi \geq 0} \left(\frac{1}{2} \log(|z - a|^2 + \xi^2) + \frac{1}{2} \log(|z + a|^2 + \xi^2) - \frac{\xi^2}{\tau} \right). \quad (54)$$

The maximal value, similarly to the previous example, depends on whether one takes into account points inside (upper line in the equation below) the support or outside (lower).

$$u(z, \tau) = \begin{cases} -\frac{1}{2} + \frac{1}{\tau} (|a|^2 + |z|^2) - \frac{1}{2} \sqrt{1 + 4(z\bar{a} + a\bar{z})^2} + \frac{1}{2} \log \left(\frac{\tau^2}{2} + \frac{1}{2} \tau \sqrt{\tau^2 + 4(z\bar{a} + a\bar{z})^2} \right) \\ \log |z^2 - a^2| \end{cases} \quad (55)$$

An application of the Poisson law yields the spectral density:

$$\rho(z) = \frac{\tau|a|^2}{2\pi (z\bar{a} + a\bar{z})^2 \sqrt{4(z\bar{a} + a\bar{z})^2 + \tau^2}} - \frac{|a|^2}{2\pi (z\bar{a} + a\bar{z})^2} + \frac{1}{\pi\tau}. \quad (56)$$

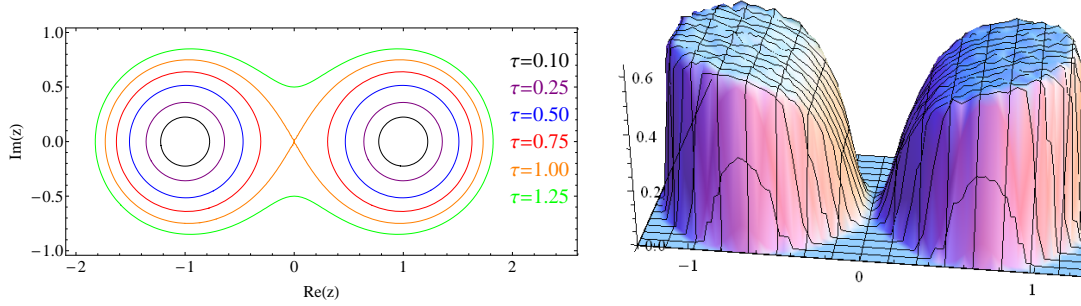


Figure 5: Evolution of the support of the diffusing deterministic matrix of the parameter $a = 1$ (left). 3D plot of the eigenvalue density at the moment when the support connects for the parameter $a = 1/\sqrt{2}$ (right).

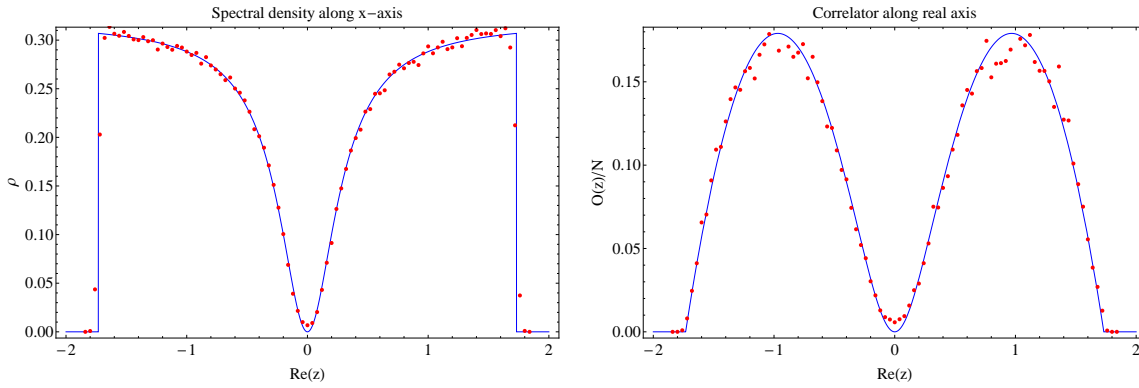


Figure 6: The spectral density and the correlator along real axis at the critical time $\tau = 1$. Numerical data were obtained by a diagonalization $1.2 \cdot 10^4$ matrices of size $N = 500$.

4.3 Band matrix

Hitherto we considered normal matrices that commute with their Hermitian adjoint. This time we break the symmetry letting the band just above the diagonal be non-zero. The initial entries are either zero or constant, therefore the matrix can be written as $(\mathbb{X}_0)_{ij} = \alpha \delta_{i+1,j}$. At the beginning, the eigenvalue density is the same as in the Ginibre case, however sets of eigenvectors differ, as it will be seen later, this disparity leads to different results.

The initial determinant can be derived with the use of the recursion formula for Toeplitz matrices, we present here the result:

$$D_0 = \begin{vmatrix} a & b & 0 & \dots & 0 \\ \bar{b} & a & b & & 0 \\ 0 & \bar{b} & a & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b \\ 0 & \dots & 0 & \bar{b} & c \end{vmatrix} = \frac{1}{\Delta} \left(c(a_+^N - a_-^N) - |b|^2(a_+^{N-1} - a_-^{N-1}) \right) \quad (57)$$

With the notation $a = |z|^2 + |\alpha|^2 + r^2$, $b = -\alpha \bar{z}$, $c = |z|^2 + r^2$, $\Delta = \sqrt{a^2 - 4|b|^2}$ and $a_{\pm} = (a \pm \Delta)/2$, where N is a size of the matrix. We remark that the radicand in Δ is positive except the points in the parameter space where $r = 0$, $|z| = |\alpha|$. In that case the formula (57) is not valid, since the characteristic equation in the recurrence has a double root. We turn back to the definition of D (14) and calculate explicitly $D_0 = \det(z\mathbb{1} - \mathbb{X}_0)(\bar{z}\mathbb{1} - \mathbb{X}_0^\dagger) = (z\bar{z})^N$. This condition leads to the eigenvalue spectrum given by a delta distribution located in 0. For any other points in order to find u_0 we approximate

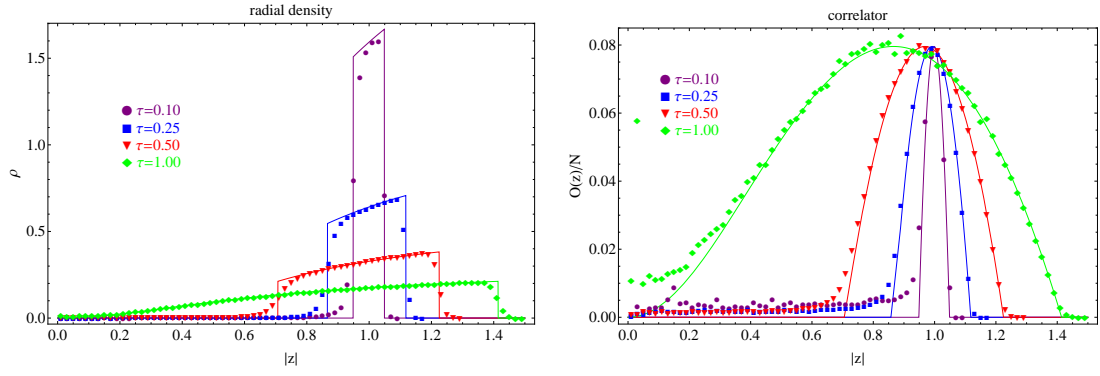


Figure 7: The eigenvalue density and the correlator in a radial variable for diffusing band matrix. Data has been gathered by diagonalizing $3 \cdot 10^3$ matrices of size $N = 1000$ with $\alpha = 1$. The solid line represents the theoretical formula, discrepancies are finite size effects, smoothing the spectrum.

it in large N limit, writing as

$$u_0 = \frac{1}{N} \log D_0 = -\frac{1}{N} \log \Delta + \log a_+ + \frac{1}{N} \log \left[c \left(1 - \left(\frac{a_-}{a_+} \right)^N \right) - \frac{|b|^2}{a_+} \left(1 - \left(\frac{a_-}{a_+} \right)^{N-1} \right) \right]. \quad (58)$$

It is clearly seen that the inequality $a_- < a_+$ holds if only $\Delta \neq 0$, therefore their ratio tends to zero when N goes to infinity. Moreover, one can easily prove that $ca_+ > |b|^2$, thus there is no singularity in the logarithm. Finally, in the $N \rightarrow \infty$ limit, only one term survives:

$$u_0 = -\log 2 + \log \left[|z|^2 + |\alpha|^2 + r^2 + \sqrt{(|z|^2 - |\alpha|^2)^2 + r^4 + 2r^2(|z|^2 + |\alpha|^2)} \right]. \quad (59)$$

The initial condition in v takes the form of

$$|v_0| = \frac{r}{\sqrt{|\alpha|^4 + 2|\alpha|^2(r^2 - |z|^2) + (r^2 + |z|^2)^2}}. \quad (60)$$

Characteristics are similar to the two previously studied case. By the standard procedure which relies on putting $r = 0$ in the eq. (40) with eq. (41) fulfilled simultaneously we determine the time when the shock appears, that is:

$$\tau = \left| |\alpha|^2 - |z|^2 \right|. \quad (61)$$

This equation represents two circles of radii $r_{\pm} = \sqrt{|\alpha|^2 \pm \tau}$ - the boundary of the eigenvalue support. Initially, the area where eigenvalues appear is confined to the annulus given by these two radii, at time $\tau = |\alpha|^2$ the inner radius vanishes and the support becomes a simply-connected set.

The eq. (42) simply yields the correlator:

$$O(z) = \frac{N}{\pi} |v|^2 = \begin{cases} \frac{N}{\pi \tau^2} \left(\sqrt{4|\alpha|^2 |z|^2 + \tau^2} - |\alpha|^2 - |z|^2 \right) & \text{inside annulus} \\ 0 & \text{outside} \end{cases} \quad (62)$$

With no detailed calculations we provide the formula for spectral density valid within the support

$$\rho(z) = \frac{1}{\pi \tau} \left(1 - \frac{|\alpha|^2}{\sqrt{\tau^2 + 4|z|^2 |\alpha|^2}} \right). \quad (63)$$

Both these result were confirmed by numerical simulations, as it was shown in the fig. 7.

5 Summary

The newly discovered dynamic in diffusing matrices, presented in this paper, turned out to be nonlinear, directing to the Burgers' equation. The properties of the Burgers' evolution are present in the correlator where the appearance of the shock in the parameter space determines the eigenvalue spectrum.

Derived equations allow one to readily find the spectral density, the correlator and the borderline of the support, which makes the Burgers' analysis a powerful technique in studying diffusing matrices, perhaps simpler in applications than the algebraic way, although not general.

The presented work is only a tiny part in the wide field of random matrices. This does not close the problem since there are still many open questions. As one can see the analytical approach gives exactly the same results as the algebraic one, so probably there exists a one-to-one correspondence between these two paths which has not been remarked so far.

During the interpretation of the introduced objects the order of integration and taking a limit $r \rightarrow 0$ was reversed which may not be always true, in particular for the ensembles whose eigenvalue density is singular but still integrable. The task for the future work is to study such matrices and precisely define when such an operation is justified. The number of examined random walks is limited, there are many other classes besides the Gaussian diffusion that may exhibit a new dynamic.

6 Acknowledgments

The author would like to express his sincere gratitude to prof. Maciej A. Nowak for introducing into this topic of research and to his collaborators prof. Zdzisław Burda, Jacek Grela and Piotr Warchoł for the discussion and their precious remarks.

This research is supported by the Grant DEC-2011/02/A/ST1/00119 of the National Centre of Science and by PL-Grid Infrastructure.

Bibliography

- [1] E. Gudowska-Nowak, R. A. Janik, J. Jurkiewicz, M. A. Nowak, *Nucl. Phys. B* **670**, 479 (2003).
- [2] J.-P. Blaizot, M. A. Nowak, P. Warchoł, *Phys. Rev. E* **87**, 052134 (2013).
- [3] T. Cabanal Duvillard, A. Guionnet, *Ann. Prob.* **29**, 1205 (2001).
- [4] J.-P. Blaizot, M. A. Nowak, *Phys. Rev. E* **82**, 051115(2010).
- [5] J.-P. Blaizot, M. A. Nowak, P. Warchoł *Phys. Rev. E* **89**, 042130
- [6] Z. Burda, J. Grela, M. A. Nowak, P. Warchoł, W. Tarnowski *To be published* arXiv: 1403.7738
- [7] F. Dyson, *J. Math. Phys.* **3**, 140-156, (1962)
- [8] R. A. Janik, M. A. Nowak, G. Papp, I. Zahed *Nucl. Phys. B* **501**, 603 (1997)
- [9] J. Ginibre, *J. Math Phys* **6**, 440 (1965)
- [10] A. Jarosz, M. A. Nowak, *J. Phys A* **39**, 10107 (2006)
- [11] D. V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables*, (Am. Math. Soc., Providence, RI, 1992).
- [12] J. T. Chalker, B. Mehlig, *Phys. Rev. Lett.* **81**, 3367 (1998).
- [13] R. A. Janik, W. Noerenberg, M. A. Nowak, G. Papp, I. Zahed, *Phys. Rev. E* **60**, 2699 (1999).
- [14] W. Tarnowski *Free Random Variable approach to additive diffusion on matrices* Sophomore Thesis (unpublished) (2013)
- [15] E. Hopf, *J. Math. Mech.* **14** (1965), 951-973
- [16] P. D. Lax, *Comm. Pure Appl. Math.* **10**(1957), 537- 566.
- [17] F. A. Berezin, *Introduction to superanalysis*. D. Reidel Publishing Co., Dordrecht, 1987.

Appendix A Grassmann algebra

In order to describe fermionic fields in a path integral formalism, one introduces [17] a collection of independent variables which anticommute with each other but commute with arbitrary complex numbers.

$$\theta_i \theta_j = -\theta_j \theta_i, \quad \forall z \in \mathbb{C} \quad \theta_i z = z \theta_i. \quad (64)$$

In particular, square of any Grassmann number vanishes. Elements of this algebra can also be integrated. Grassmann integrals are linear

$$\int [af(\theta) + bg(\theta)] d\theta = a \int f(\theta) d\theta + b \int g(\theta) d\theta \quad (65)$$

and satisfy the partial integration formula:

$$\int \left(\frac{d}{d\theta} f(\theta) \right) d\theta = 0. \quad (66)$$

Basic integrals are as follows:

$$\int 1 d\theta = 0, \quad \int \theta d\theta = 1. \quad (67)$$

A crucial remark is that the order of variables in the integrand is essential, for example

$$\int \int \theta_i \theta_j d\theta_i d\theta_j = - \int \theta_j \int (\theta_i d\theta_i) d\theta_j = - \int \theta_j d\theta_j = -1. \quad (68)$$

Functions of Grassmann numbers are determined by their series expansions which are truncated due to the vanishing square. For instance, an exponent:

$$e^{\theta_i} = 1 + \theta_i \quad (69)$$

Let θ be a column of different Grassmann variables θ_i , $i = 1, \dots, N$, then θ^T is the row of the same variables. For any $N \times N$ matrix \mathbb{A} an equality holds:

$$\det \mathbb{A} = \int \exp(\theta^T \mathbb{A} \eta) \prod_{i=1}^N d\theta_i d\eta_i. \quad (70)$$

After expanding the exponent in a series, term of order higher than n vanishes since at least one variable has to appear twice, terms of lower order vanishes after the integration because of (67). Only these terms of a power of n survive which consist of distinct variables and due to (68) the sign in front of each term is equal to the sign of a permutation of the matrix elements, that immediatly leads to the determinant.