Department of Physics, Astronomy and Applied Computer Science

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Various aspects of non-perturbative dynamics of gauge theory and the AdS/CFT correspondence

presented by

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Abstract

The Thesis studies real-time physics of certain strongly coupled planar gauge theories using a dual gravitational description. The dynamics of interest is the boost-invariant flow in the setting of holographic conformal field theories in 3+1 dimensions with dual description in terms of Einstein gravity in 5-dimensional asymptotically anti-de Sitter spacetime. Resulting equations of motion are solved analytically at late and early proper time. The late-time gravity solution, which is dual to boost-invariant hydrodynamics, is shown to be regular contrary to previous claims and its causal structure is analyzed with possible implications on generalizations of entropy to time-dependent field theory configurations. Furthermore, different scenarios in the proposal to make quantitative comparisons between strongly coupled quark-gluon plasma and holographic descriptions of conformal field theory are examined by analyzing the form of corrections to certain transport coefficients appearing in second order hydrodynamics from higher curvature terms in the dual gravity theory. The far-from-equilibrium dynamics of conformal plasma is studied in the regime of early proper time and it is shown, in contrast with the late-time expansion, that a scaling solution does not exist. Gauge theory dynamics in this regime depends on initial conditions encoded in the bulk behavior of metric coefficients at some initial proper time. The relation between the early-time expansion of the energy density and initial conditions in the bulk of anti-de Sitter space time is provided. Further investigations reveal rich, initial conditions dependent far-from-equilibrium dynamics. The impact of this study on the problem of thermalization at strong coupling is discussed.

Abstrakt

Praca porusza zagadnienia dynamiki silnie sprzężonych holograficznych teorii cechowania w granicy dużej liczby kolorów przy użyciu dualnego opisu grawitacyjnego. W szczególności rozważana jest boost-niezmiennicza ekspansja plazmy konforemnych teorii cechowania w 3+1 wymiarach o dualnym opisie w języku grawitacji Einsteina w 5-wymiarowych czasoprzestrzeniach asymptotycznie anty-de Sittera. Otrzymane równania ruchu rozwiązane są analitycznie w granicy dużych i małych czasów. Pokazane zostaje, że rozwiązanie grawitacyjne dla dużych czasów opisujące boost-niezmienniczą hydrodynamikę jest regularne w sensie cenzury kosmicznej. Rozważania dotyczące termodynamiki tego rozwiązania w języku kwazilokalnych horyzontów prowadzą do fenomenologicznej definicji entropii czarnych bran i mogą mieć znaczenie dla uogólnienia pojęcia entropii na procesy bliskie równowagi w teorii cechowania. Zbadany zostaje także wpływ wiodących poprawek wyższych rzędów w krzywiznach do działania grawitacyjnego na wartości współczynników transportu drugiego rzędu w holograficznych teoriach cechowania oraz przeanalizowane zostają różne scenariusze w ramach których takie poprawki mogą się pojawić. W pracy rozwiązano także analitycznie problem dynamiki grawitacyjnej dla małych czasów, która odpowiada silnie nierównowagowej fizyce teorii cechowania, a także omówiono uzyskane wyniki w kontekście problemu termalizacji w silnie sprzężonych teoriach cechowania.

Contents

Foreword

As a Ph.D. student during the period of October 2007 - May 2010, I had the joy to take part in a rapid development of the field called *applied gauge/gravity duality*. It is an interdisciplinary field of theoretical physics, which applies tools provided by the string theory to study strongly coupled setups and toy-models inspired by the real-world physics (for reviews see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9]). The primary motivation for undertaking that path of research is physics of quark-gluon plasma, which above, but not far above, the critical temperature is a strongly coupled phase of Quantum Chromodynamics. This new form of matter has been under extensive experimental studies in the Relativistic Heavy Ion Collider and will be also produced and probed at the Large Hadron Collider. In the absence of robust methods to calculate dynamical properties of strongly coupled QCD, gauge/gravity duality offers an unique opportunity to learn qualitative lessons about real-time physics of certain gauge theories at strong coupling. In general, it is hard to judge how important these developments will be for future understanding of QCD itself. However, so far there have been several important lessons (with experimental implications), which followed directly or indirectly from this line of research

1. Obtaining concrete values of transport properties of certain strongly coupled gauge theories [10];

2. Understanding that small ratio of shear viscosity to entropy density might be correlated with the strongly coupled physics [11];

3. Finding missing terms in second order conformal hydrodynamics [10, 12, 13];

4. Better understanding of hydrodynamics of theories with anomalies, which might lead to discovering new effects in quark-gluon plasma [14, 15].

Most of the results presented in this Thesis were published in the articles listed below. An executive summary of the most important achievements is given in the concluding Chapter.

1. P. Benincasa, A. Buchel, M. P. Heller and R. A. Janik, "On the supergravity description of boost invariant conformal plasma at strong coupling," Phys. Rev. D **77**, 046006 (2008) [arXiv:0712.2025 [hep-th]];

2. M. P. Heller, P. Surowka, R. Loganayagam, M. Spalinski and S. E. Vazquez, "Consistent Holographic Description Of Boost-Invariant Plasma," Phys. Rev. Lett. **102**, 041601 (2009);

3. G. Beuf, M. P. Heller, R. A. Janik and R. Peschanski, "Boost-invariant early time dynamics from AdS/CFT," JHEP **0910**, 043 (2009) [arXiv:0906.4423 [hep-th]];

4. A. Buchel, M. P. Heller and R. C. Myers, "sQGP as hCFT," Phys. Lett. B **680**, 521 (2009) [arXiv:0908.2802 [hep-th]];

5. I. Booth, M. P. Heller and M. Spalinski, "Black brane entropy and hydrodynamics: the boost-invariant case," Phys. Rev. D **80**, 126013 (2009) [arXiv:0910.0748 [hep-th]].

During three years which span the period of publication of the original works there has been an enormous progress in the field. This Thesis takes the perspective of the mature field, which eventually applications of gauge/gravity duality have developed into, rather than directly the views presented in the original publications. In particular, the large proper time limit of boost-invariant flow is presented as an example of fluid/gravity duality [10] (see [6] for a review), rather than an independent phenomenon. Moreover, the initial results of [16, 17] suggesting an inconsistency of the gravity dual to the boost-invariant flow are reinterpreted here following the results of [18] as a mere failure of the particular coordinate chart in describing the perfectly regular gravity dual. Apart from that, the universality of transport properties of holographic conformal field theories with a classical gravity dual is now understood as a feature of the planar limit and strong 't Hooft coupling. In particular, the shear viscosity of certain holographic gauge theories [19] violates (though mildly) the famous conjectured bound $\eta/s \geq 1/4\pi$ in natural units [11]. As a result, the question whether there is a physical bound on the dissipation in the systems is still open (see [20] for a review).

There are numerous people who influenced directly or indirectly the contents of the Thesis. In the very first place I would like to thank my advisor, Romuald A. Janik, who introduced to me and has guided me through the fascinating subject of gauge/gravity duality and its applications. I am also very indebted to Michał Spaliński for various discussions, advice and fruitful collaborations. Moreover, I would like to thank Alex Buchel and Rob Myers for inviting me to the Perimeter Institute and teaching me a lot of good physics. Last but not least I would like to acknowledge discussions with friends, colleagues, collaborators and masters, most notably Ofer Aharony, Ivan Booth, Paul Chesler, Hong Liu, Robi Peschanski, Shiraz Minwalla, Mukund Rangamani, Dam Son, Andrei Starinets and Larry Yaffe.

The author is greatful to Romuald A. Janik and Michał Spaliński for valuable comments on the draft. Most of the results presented in this Thesis rely on `diffgeo.m`, Matthew Headrick's excellent `Mathematica` package for tensorial manipulations. Figures 5.1 and 5.2 are taken from the preprint version of [21] by Larry Yaffe and Paul Chesler.

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Introduction

One of the most important challenges in contemporary physics is understanding quantum field theories at non-perturbative level. There are various motivations to undertake that path of research, majority of them tied to the success of quantum field theory framework as a(n effective) description of microscopic phenomena in high energy and condensed matter physics. The new motivation to deal with quantum field theories is the AdS/CFT correspondence or more generally gauge/gravity duality [22, 23, 24]. This conjecture relates particular quantum field theories to certain vacua of string theory stating complete equivalence (in a sense of two languages describing the same physics). The AdS/CFT duality is the first concrete example of holographic correspondence proposed by Susskind and 't Hooft [25, 26], since it relates certain quantum field theories with gauge symmetry in lower dimensional non-dynamical spacetimes (i.e. 3+1 dimensional) to 10-dimensional string theory or 11-dimensional M-theory solutions. The correspondence is a weak/strong coupling duality, which means that strongly coupled field theory is equivalent to the weakly coupled string theory description and vice versa[1].

Gauge/gravity duality is a conjecture: its weak/strong coupling character makes it very difficult to prove. However, a highly suggestive amount of evidence has been gathered during the years, all in support of the correspondence[2]. This makes gauge/gravity duality a conservative statement and any disagreement between the two sides of the conjectured equivalence would imply a serious gap in the current understanding of quantum field theories or string theory.

Assuming that the AdS/CFT correspondence is correct opens a new exciting possibility of studying *real-time* non-perturbative physics of certain (dubbed holographic) quantum field theories. More concretely, the correspondence maps non-perturbative physics of those theories at large number of colors to, in principle, solvable problems in classical gravity. This means that higher dimensional Einstein gravity supplemented with necessary or desired matter fields is capable of describing a wide range of quantum field theory phenomena (including real-time physics) in the dual geometric language.

Among all quantum field theories, the one of particular interest is Quantum Chromodynamics. This theory has been recently probed experimentally at high energies and densities in the collisions of heavy ions at Relativistic Heavy Ion Collider [1][3]. The collisions of heavy ions are highly dynamical and complicated processes with both perturbative and non-perturbative

[1]There is also an intermediate regime where both sides of the correspondence are complicated quantum theories.

[2]Some are the following: matching between Kaluza-Klein modes of Type IIB supergravity on $AdS_5 \times \mathbf{S}^5$ and the chiral operators of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions [23]; success of the holographic renormalization program – an agreement between the UV structure of correlation functions on both sides of the correspondence [27] (the author thanks Kostas Skenderis for pointing this out); agreement between string theory [28] and perturbative gauge theory calculation [29] of Konishi operator at four loops.

[3]The heavy ion collisions program is going to be an important part of LHC agenda as well.

physics involved. A particularly interesting outcome of the experiment was that the late time physics of nuclear matter excited in the collision is well described by an almost ideal hydrodynamics. This experimental observation raises questions why nuclear matter thermalizes quickly (applicability of hydrodynamics) and why dissipative effects are small (small viscosity). These features of the collective flow of a small (of order of nucleus size) portion of quark-gluon plasma are nowadays attributed to non-perturbative effects. String theory methods provided the first calculation of the shear viscosity in certain strongly coupled quantum field theories [30]. This result, although not directly applicable to QCD itself, played an important role in the paradigm shift from perturbative QCD with very large shear viscosity to strongly coupled quark-gluon plasma being one of the most perfect fluids in nature [1].

While the famous 't Hooft argument [31] suggests that QCD has a dual description in terms of string theory, gauge/gravity duality has not been (yet) formulated for QCD itself. This however did not stop the string theory research in trying to understand the implications of strong coupling on real-time physics of gauge theories with a view towards QCD. In particular, although the vacua of strongly coupled theories with a classical gravity dual differ significantly from the QCD vacuum, there are suggestive qualitative features shared by the holographic plasmas and QCD above, but not far above critical temperature, being precisely a range of temperatures achieved at RHIC[4]. The (quasi)conformality [3], Debye screening of color charge [32] and small shear viscosity of theories with classical gravity duals, which are also features of QCD in the temperature range achieved at RHIC, might suggest to use holographic techniques in order to study qualitative features of strongly coupled plasmas and apply certain outcomes of this program to experimental investigations. This led to an extensive study of QCD-inspired setups using the gauge/gravity duality with some concrete successes of the approach listed in the Foreword.

The program of applications of gauge/gravity duality to study QCD-inspired setups has some natural limitations. One technical obstacle is that the gravity dual language is tractable only when quantum (string worldsheet and string loop) effects are negligible. This amounts to studying strongly coupled holographic gauge theories in the planar limit. Moreover, including string theory effects in an effective low-energy gravity action can be done in a self-consistent fashion only when higher derivative corrections are treated perturbatively[5]. Some of higher curvature contributions correspond to finite coupling corrections on the field theory side and their perturbative treatment on top of the two-derivative background amounts to staying within the strong (but in that case finite) coupling planar gauge theory on the dual side. On the other hand, certain aspects of the evolution of RHIC fireball are believed to be governed by perturbative processes and their AdS/CFT description will not give a reliable qualitative picture (although then it might be unnecessary). This means that gauge/gravity correspondence might give some direct or indirect hints about the dynamics of QCD plasma only if the latter is strongly coupled and departures from planar limit are not crucial.

This Thesis develops real-time gravitational methods within the AdS/CFT correspondence focusing on a particular example of QCD-inspired dynamics given by the boost-invariant flow [33] and its gravity dual[6]. The boost-invariant dynamics of interest is a very simple one-dimensional expansion of plasma with additional symmetries of boost-invariance and rotational invariance

[4] i.e. temperatures of order 350 MeV with deconfinement temperature being 170 MeV.

[5] With a notable exceptions of Gauss-Bonnet and Lovelock gravities.

[6] Both conformality on the field theory side and boost-invariant character of its dynamics are chosen because of simplicity.

along the expansion axis, as well as translational invariance in the perpendicular plane. Although such boost-invariant flow is not a realistic approximation for the dynamics of the quark-gluon plasma at RHIC (thus will not lead to results which can be compared with experimental data), it is still able to capture some of the physics of interest[7]. The reason for focusing on the boost-invariant example is that gravitational calculations in AdS/CFT correspondence are performed in at least one more spatial coordinate than on the field theory side. This inquires that any time-dependence in holographic quantum field theory requires solving Einstein equations, which in the simplest dynamical setup are a system of partial differential equations for a couple of functions depending on at least two variables (time + radial coordinate in AdS). On the other hand, late-time behavior governed by the universal hydrodynamic tail requires apart from temporal also spatial gradients on the field theory side (otherwise hydrodynamic modes are not excited). This, in most cases, introduces dependence on additional variables on the gravity side and makes the string theory calculation very demanding. However, the boost-invariant example is a notable exception. The assumption of boost-invariance mixes the spatial and temporal gradients in such a way, that quantum field theory observables depend on a single coordinate – proper-time. This makes the gravity dual tractable using analytic methods both at late and early times with the relatively simple numerics providing results about the dynamics at transient times.

The structure of the Thesis is the following. Chapters 1-3 review theoretical background, whereas Chapters 4-7 present original results. Chapter 1 provides a short exposure to the methods of gauge/gravity duality with an emphasis put on applications. Chapter 2 reviews a modern treatment of conformal relativistic hydrodynamics, whereas Chapter 3 introduces the boost-invariant model of field theory dynamics. The gravity dual to boost-invariant hydrodynamics is presented in Chapter 4 with global analysis of the resulting space-time postponed to Chapter 5. Going beyond the supergravity paradigm in the holographic picture of hydrodynamics is a subject of Chapter 6. Chapter 7 concerns far-from-equilibrium boost-invariant dynamics. The results are summarized in the last part of the Thesis.

[7] In particular, its late time dynamics is governed by hydrodynamic tail, whereas early time dynamics by far-from-equilibrium physics.

Chapter 1

The Gauge/Gravity duality

1.1 Holographic dictionary

The gauge/gravity duality is a conjectured, but well motivated and tested, *exact equivalence*[1] between certain gauge theories and string theory solutions. Gauge theories, which have a string theory description, are called holographic, since the dual dynamics involves more spacetime dimensions than the quantum field theoretic one and one can think of those quantum field theories as "holograms" of string theory physics. Although the duality is conjectured to hold for certain conformal and confining theories in flat or curved backgrounds of various dimensionality, with various matter content, with or without supersymmetry, with various gauge groups and at arbitrary coupling, this Thesis studies QCD-inspired setups in holographic conformal field theories (hCFTs) in planar strongly coupled limit in $(3+1)$-dimensional Minkowski spacetime, the primary reason being simplicity. The masterfield description of complicated quantum dynamics of those gauge theories is given in terms of type IIB supergravity solutions, being the low-energy limit of type IIB string theory [34], on product of 5-dimensional[2] asymptotically anti-de Sitter spacetime and 5-dimensional (compact) Einstein manifold[3]. In all applications covered in this Thesis compact manifold will not be excited and one can perform Kaluza-Klein reduction leaving only zero modes. The latter have a consistent truncation to an universal[4] gravity action [35] consisting of 5-dimensional Einstein-Hilbert term supplemented with a negative cosmological constant

$$I_{gravity} = \frac{1}{2l_P^3}\int_M \mathrm{d}^5x \left\{\mathcal{R} + \frac{12}{\mathcal{L}^2}\right\}, \tag{1.1}$$

where l_P is the 5-dimensional Planck length and $\mathcal{L}$ is the curvature radius of anti-de Sitter spacetime[5]. For $\mathcal{N}=4$ super Yang Mills theory with number of colors N_c and 't Hooft coupling λ, the first entries in the holographic dictionary take the form

$$\lambda = \mathcal{L}^4/\alpha'^2 \;\text{ and }\; g_{YM}^2 = \lambda/N_c = 4\pi g_s, \tag{1.2}$$

[1] Any holographic gauge theory phenomenon should have a dual string theory counterpart and vice versa.

[2] Note, that on the gauge theory side the dimensionalities appear usually in the "$(3+1)$-form" to stress real-time character of the problems covered in this Thesis. Although the string theory side dimensionalities are expressed in the standard form (i.e. "5-dimensional", "10-dimensional"), all bulk new developments covered in the rest of the text required Minkowski signature.

[3] An example of such compact manifold is a 5-sphere.

[4] Holding for infinitely many holographic conformal field theories in 3+1 dimensions, including $\mathcal{N}=4$ super Yang Mills, for which AdS/CFT correspondence was initially postulated.

[5] The most symmetric solution of Einstein's equations with negative cosmological constant.

where string α' parameter is related to fundamental string tension and g_s is the string coupling constant. This formula makes it clear that the planar limit on the gauge theory side is dual to tree level string theory, whereas keeping 't Hooft coupling large decouples massive string states leaving only the supergravity multiplet. For an extensive discussion of the holographic dictionary in more complicated examples of gauge/gravity duality see [36].

The (Poincare patch of) anti-de Sitter spacetime, which is identified with vacuum state of dual conformal field theory[6], in Fefferman-Graham coordinates takes the form[7]

$$\mathrm{d}s^2 = G_{AB}\,\mathrm{d}x^A \mathrm{d}x^B = \frac{\mathcal{L}^2}{z^2}\left\{\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right\} \tag{1.3}$$

with $\eta_{\mu\nu}$ being the $(3+1)$-dimensional Minkowski metric, z the radial direction in anti-de Sitter spacetime running from 0 to ∞ and again $\mathcal{L}$ its curvature radius[8]. AdS vacuum metric (1.3) is symmetric with respect to dilatations, namely simultaneous rescalings of x^μ and z coordinates

$$x^\mu \to \alpha\, x^\mu \;\text{ and }\; z \to \alpha\, z. \tag{1.4}$$

This suggests the interpretation of the radial direction in AdS as an energy scale in dual gauge theory. UV physics of gauge theory should be related to the behavior of the asymptotically AdS metric at small z, whereas large z behavior governs the IR part of gauge theory dynamics. Because of $1/z^2$ warping, AdS spacetime is a throat-like geometry with (conformal [23, 37]) boundary located at $z = 0$. In order to see that AdS spacetime indeed has a boundary, one can look at the equation for radial null geodesics, which takes the form

$$\mathrm{d}s^2 = 0 = \tfrac{1}{z^2}\left\{-\mathrm{d}t^2 + \mathrm{d}z^2\right\}. \tag{1.5}$$

Solutions of this equation are given by $z = z_0 \pm t$, which implies that null geodesics reach surface $z = 0$ in finite coordinate time. This means that one indeed needs to specify boundary conditions at $z = 0$ for all fields in the gravitational theory, including 5-dimensional metric tensor itself. The boundary condition for the 5-dimensional metric has an interpretation of a metric in which the dual quantum field theory is formulated and in (1.3) is chosen to be the Minkowski metric $\eta_{\mu\nu}$. It needs to be stressed that the vacuum AdS metric is an exact solution of Einstein's equations. There are two interesting directions of research to pursue at this point. The first is to consider dynamical solutions of (1.1) with Minkowski metric taken as a boundary condition, and such studies using a very specific example of holographic quantum field theory dynamics – the boost-invariant flow (see Chapter 3 and references therein for an introduction) – are the main subject of the Thesis. The second interesting and recently revived avenue is to construct ground or thermal states of strongly coupled planar gauge theories on curved manifolds using the gravitational prescription. In particular, one can consider more involved situations, in which the field theory is put on some non-dynamical curved background (e.g. asymptotically flat Schwarzschild black hole in 3+1 dimensions [38]), which is at the same time interpreted as a boundary condition for a 5-dimensional asymptotically *locally* AdS metric. In such cases the metric (1.3) with $\eta_{\mu\nu}$ literally replaced by a metric on some curved manifold $m_{\mu\nu}(x)$ is no longer an exact solution of Einstein's equations, but it is an approximate solution

[6] More precisely the gravity dual to vacuum state at strong coupling and large number of colors is full 10-dimensional metric $AdS_5 \times M^5$.

[7] Note that capital Latin indices denote 5-dimensional (bulk) coordinates, whereas Greek indices 4-dimensional ones (coordinates on slices of constant radial variable).

[8] $\mathcal{L} \gg l_P$ for consistency of classical gravitational description.

near $z = 0$. That part of holography is not explored in the Thesis, but approximate methods developed in original publications [18, 39] presented in Chapters 4 and 7 based on pioneering approach introduced by Janik and Peschanski in [40] should be applicable as well to the case of a curved boundary metric depending on a single time-like coordinate (see also [41])[9].

Excitations on top of the vacuum can be studied systematically by solving Einstein's equations in the near-boundary (small z) expansion. The most general (not assuming any symmetries) metric Ansatz in the Fefferman-Graham chart takes the form

$$\mathrm{d}s^2 = \frac{\mathcal{L}^2}{z^2}\left\{\mathrm{d}z^2 + g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right\}, \tag{1.6}$$

where the 4×4 matrix $g_{\mu\nu}$ is a function of both x^μ ("gauge theory"[10]) directions as well as the radial coordinate z and for $z = 0$ reduces to a metric in which the dual gauge theory is formulated. The presence of a boundary implies that the variational principle for the action (1.1) is ill-posed and has to be supplemented with a boundary term, which is the standard Gibbons-Hawking term [42][11]

$$I_{GH} = \frac{1}{l_P^3}\int_{\partial M}\mathrm{d}^4x\sqrt{-\det g^{(\mathrm{ind})}}K. \tag{1.7}$$

Here K is the trace of the extrinsic curvature defined as

$$K^{AB} = \frac{1}{2}P^{AC}P^{BD}\left(\nabla_C n_D + \nabla_D n_C\right), \tag{1.8}$$

where

$$\begin{aligned} P^{AB} &= G^{AB} - n^A n^B, \\ n^A &= -\frac{z}{\mathcal{L}}\left[\partial_z\right]^A \end{aligned} \tag{1.9}$$

with the latter choice tied to Fefferman-Graham coordinates and g_{ind} being the 4-dimensional metric induced on the boundary[12].

It is matter of direct calculation to show that Einstein's equations for flat boundary metric are solved by

$$g_{\mu\nu} = \eta_{\mu\nu} + 0\cdot z^2 + g^{(4)}_{\mu\nu}(x)\, z^4 + \ldots, \tag{1.10}$$

where the expansion contains only even powers of z, the z^2 term vanishes and $g^{(4)}_{\mu\nu}(x)$ is an arbitrary 4×4 matrix which is conserved and traceless[13]

[9] An example of such setup might be some cosmological, highly symmetric metric taken as a boundary condition – the author thanks Alex Buchel for discussions on that point.

[10] Note that in Fefferman-Graham chart x^μ have an interpretation of coordinates on the boundary.

[11] The presence of Gibbons-Hawking (1.7) term has important consequences for obtaining expectation value of gauge theory energy-momentum tensor.

[12] In Fefferman-Graham chart it is taken to be $g^{(\mathrm{ind})}_{\alpha\beta} = \mathcal{L}^2/z^2\, g_{\alpha\beta}\big|_{z=\xi}$ with regulator ξ taken to 0 at the end of calculation.

[13] Note the presence of covariant derivative – although the boundary is taken to be Minkowski spacetime, coordinates on boundary might be curvilinear (this is the case in the rest of the Thesis, where gravity dual of the boost-invariant flow is considered).

$$\nabla^\mu g^{(4)}_{\mu\nu}(x) = 0 \quad \text{and} \quad \eta^{\mu\nu} g^{(4)}_{\mu\nu}(x) = 0. \tag{1.11}$$

Terms higher order in z turn out to be fully specified by $g^{(4)}_{\mu\nu}(x)$ and its derivatives [37, 40][14]. Both tensorial structure with respect to boundary coordinates and properties of conservation and tracelessness (1.11) strongly suggest to regard $g^{(4)}_{\mu\nu}(x)$ as being proportional to energy-momentum tensor of boundary gauge theory (or more correctly its one-point function, since boundary theory is quantum mechanical). In order to make it precise one needs to evaluate expectation value of the energy-momentum tensor of holographic conformal field theory in terms of dual gravity action and compare the result with $g^{(4)}_{\mu\nu}(x)$.

The supergravity action, a reduction of which is the universal gravity action (1.1), is a saddle point of the path integral representation of the string theory partition function Z_{string} on $\mathrm{AdS}_5\times\mathrm{M}^5$. At the core of the AdS/CFT correspondence lies the identification of Z_{string} with the holographic gauge theory partition function Z_{gauge}. The gauge theory generating functional for connected correlation functions of the energy-momentum tensor is given by $\log Z_{gauge}[m_{\mu\nu}]$, where the background metric $m_{\mu\nu}$ is understood as a source for the energy-momentum tensor. One-point function of the energy-momentum tensor for a gauge theory in a background metric $\eta_{\mu\nu}$ is then defined by

$$\left\langle T^{\alpha\beta}\right\rangle = -\frac{2\,i}{\sqrt{-m}}\,\frac{\delta}{\delta\, m_{\alpha\beta}} \log Z_{gauge}[m_{\mu\nu}]\Big|_{m_{\mu\nu}=\eta_{\mu\nu}}. \tag{1.12}$$

The identification of partition functions suggests for the case of planar strongly coupled gauge theory to evaluate the saddle point contribution to the string theory partition function from the universal gravity action for an arbitrary boundary metric generalizing (1.10) and then evaluate the functional derivative

$$\left\langle T^{\alpha\beta}\right\rangle = \frac{2}{\sqrt{-m}}\,\frac{\delta}{\delta\, m_{\alpha\beta}} \left\{ I_{gravity}[m_{\mu\nu}] + I_{GH}[m_{\mu\nu}] \right\}\Big|_{m_{\mu\nu}=\eta_{\mu\nu}}. \tag{1.13}$$

Some words of caution are in order here, since the action (1.1) contains integration over the whole volume of asymptotically AdS spacetime and is formally divergent. This divergence comes from the near-boundary region of spacetime and has a holographic interpretation as the standard UV divergence on the gauge theory side. This is in line with the intuition provided by the identification of the radial direction in AdS with an energy scale on the gauge theory side. The holographic renormalization procedure proposed in [37, 27] following [44] amounts to introducing a UV regulator, so that the integration in (1.10) in the radial direction reaches $z=\xi$ instead of $z=0$, adding local covariant counter-term[15] being

$$I_{CT} = \frac{3}{l_P^3}\int_{\partial M} \mathrm{d}^4 x \sqrt{-\det g_{\text{ind}}} \tag{1.14}$$

and eventually removing the regulator by taking $\xi\to 0$ limit

$$\left\langle T^{\alpha\beta}\right\rangle = \frac{2}{\sqrt{-m^\xi}}\,\frac{\delta}{\delta\, m^\xi_{\alpha\beta}} \left\{ I^\xi_{gravity}[m^\xi_{\mu\nu}] + I^\xi_{GH}[m^\xi_{\mu\nu}] + I^\xi_{CS}[m^\xi_{\mu\nu}] \right\}\Big|_{m^\xi_{\mu\nu}=\eta_{\mu\nu}\ \text{and}\ \xi\to 0}. \tag{1.15}$$

[14] For curved background metric the near-boundary expansion is more complicated, in particular energy-momentum tensor acquires trace related to curvature of the background manifold (trace anomaly). For an extensive discussion see [43, 37].

[15] Note that counter-term does not modify the equations of motion. For curved boundary metrics another counter-term, proportional to boundary curvature, is needed.

After carefully evaluating the functional derivative including both the Gibbons-Hawking boundary term and counter-term contributions one arrives at

$$\langle T_{\alpha\beta}\rangle = \frac{2\mathcal{L}^3}{l_P^3} g^{(4)}_{\alpha\beta}, \tag{1.16}$$

which indeed justifies the previous intuition [37]. The coefficient can be computed using the holographic dictionary and in the limit of large number of colors is given by

$$\frac{2\mathcal{L}^3}{l_P^3} = \frac{N_c^2}{2\pi^2}. \tag{1.17}$$

This ratio appears as a prefactor in front of the universal gravity action (1.1) and gets very large in the planar limit, which indeed justifies taking (1.1) as a saddle point. Note also that although the result (1.17) was derived in the limit of very large 't Hooft coupling, all λ dependence has dropped out from (1.17).

It is peculiar that the equations of motion following from universal gravity action can be solved up to arbitrary order in near-boundary expansion just by providing the boundary metric and one-point function of the energy-momentum tensor. On the field theory side this feature can be attributed to the large-N_c limit and subsequent trace factorization, so that higher-point correlation functions of local gauge-invariant operators factorize to be a product of one-point functions in the leading order in N_c (see [45] for a detailed discussion). The quantum field theory dual to the universal gravity action is thus given by decoupled dynamics of one-point function of the energy-momentum tensor, which turns out to be a universal sector of dynamics for infinitely many holographic conformal field theories at large number of colors and strong coupling[16].

Holographic near-boundary reconstruction of 5-dimensional bulk metric (1.10) works for arbitrary $g^{(4)}_{\mu\nu}(x)$ obeying (1.11). It is intuitively clear that not every conserved and traceless $g^{(4)}_{\mu\nu}(x)$ will give rise to genuine dynamics of the energy-momentum tensor. In particular, it is expected that for majority of choices of this function the bulk metric will have naked singularity in the sense of curvature blow-up not covered by the event horizon. Such singularities will not be visible within the near-boundary expansion. Thus if one would like to see what is the admissible dynamics of energy-momentum tensor, one would have to solve Einstein's equations with boundary conditions (1.10) beyond power series at $z = 0$[17]. In general, this is a very difficult task, but in some cases it can be done in an approximate way, with the boost-invariant dynamics at late times being the primary example [40] (see Chapter 4 and references therein for details and further developments). On the other hand, if one adopts a more numerical GR attitude and sets regular (in the sense of cosmic censorship) initial conditions in AdS at some constant time slice, it is expected that they will give rise to a genuine, naked singularity-free evolution on the gravity side and thus to physical configuration of holographic conformal field theory (see Chapter 7 for such an approach to early time boost-invariant dynamics). Thus there is a clear interplay between cosmic censorship conjecture [46] on the gravity side and allowed dynamics of the energy-momentum tensor of holographic gauge theories.

[16] This universality is understood entirely in terms of dual gravitational picture with quantum field theory counterpart being somewhat mysterious.

[17] Note also that the Fefferman-Graham coordinates may break down in the bulk not leading to any pathologies. For the purposes of such studies one may use other, better-adapted, coordinate frames like the ingoing Eddington-Finkelstein coordinates, see Chapter 4.

1.2 Black holes and their dual interpretation

A particularly interesting situation on the gauge theory side is plasma in global equilibrium[18] described by static isotropic energy-momentum tensor

$$T^{\mu\nu} = \mathrm{diag}\,(\epsilon,\, p,\, p,\, p)^{\mu\nu}\,. \tag{1.18}$$

This form of the energy-momentum tensor with energy density and pressure related to each other via a theory-dependent equation of state does not assume anything about the type of gauge theory. There is a major simplification in the conformal case, where the equation of state is fully specified by the tracelessness condition of the energy-momentum tensor. In 3+1 dimensions it reads

$$\epsilon = 3\,p. \tag{1.19}$$

Simple dimensional analysis allows one to express energy density in terms of temperature (the only dimensionful scale present in the system[19])

$$\epsilon = e_0 T^4, \tag{1.20}$$

where e_0 is some theory-dependent constant. Imposing further the first law of thermodynamics allows to express the entropy density as a function of the dimensionless coefficient e_0 (temperature dependence alone is again fully specified by dimensional analysis)

$$s = \frac{4}{3} e_0 T^3. \tag{1.21}$$

Note that above formulas follow directly from conformal symmetry and are valid for arbitrary conformal field theory in 3+1 dimensions[20], with coefficient e_0 being theory-dependent. The energy-momentum tensor (1.18) is by definition conserved and traceless and as such can be plugged into near-boundary power series for an asymptotically AdS metric. This series can be formally resummed by solving Einstein's equations with the metric Ansatz[21]

$$\mathrm{d}s^2 = \frac{\mathcal{L}^2}{z^2}\left\{\mathrm{d}z^2 - f(z)\mathrm{d}t^2 + g(z)\mathrm{d}\vec{x}^2\right\}. \tag{1.22}$$

The outcome of this calculation, performed in [40] (see [47] for more extensive discussion), is a black brane[22] metric

$$\mathrm{d}s^2 = \frac{\mathcal{L}^2}{z^2}\left\{\mathrm{d}z^2 - \frac{\left(1-\frac{z^4}{z_0^4}\right)^2}{1+\frac{z^4}{z_0^4}}\mathrm{d}t^2 + \left(1+\frac{z^4}{z_0^4}\right)\mathrm{d}\vec{x}^2\right\} \tag{1.23}$$

with $3\,z_0^{-4} = \frac{l_P^3}{2\mathcal{L}^3}\epsilon = \frac{2\pi^2}{N_c^2}\epsilon$. The emblackening factor leads to coordinate singularity at the position of event horizon $z = z_0$ (more precisely the event horizon is located at z_0 and $t \to$

[18] Thermal state in the language of canonical ensemble.

[19] For uncharged plasma at global equilibrium different temperatures are equivalent, since they are mapped onto each other by dilatation symmetry (1.4).

[20] In particular, at arbitrary coupling and rank of gauge group.

[21] The form of metric Ansatz follows from symmetries of near-boundary expansion of bulk metric, which are determined by symmetries of CFT's energy-momentum tensor.

[22] Black brane is a black hole with planar event horizon. For metric (1.23) horizon is of the form $\mathbb{R}^{3+1}$. For an extensive discussion of black hole properties see Chapter 5 and references [48, 49].

∞), which can be avoided by choosing different coordinate chart (e.g. Eddington-Finkelstein or Kruskal coordinates). In particular, one can check that simple curvature invariants, e.g. $\mathcal{R}_{ABCD}\mathcal{R}^{ABCD}$, are regular in the vicinity of $z = z_0$. The coefficient e_0 appearing in (1.20) is hitherto unspecified, but can be fixed using self-consistency of the gravity description. So far all calculations assumed Lorenzian signature, but results can be trivially ($t \to i\, t$) continued to Euclidean signature. In particular, in thermal (Euclidean) quantum field theory the Euclidean time direction is compactified on a circle, whose circumference is identified with temperature inverse [50]. In such a setup the boundary metric is taken to be $S^1 \times \mathbb{R}^3$ and in the context of AdS/CFT such a compactification is performed both on the boundary and in the interior of Euclideanized asymptotically AdS spacetime [51]. The form of the bulk metric given by (1.23) remains unchanged apart from standard Wick rotation, but now at $z = z_0$ the Euclidean time circle shrinks to 0, which may lead to a conical singularity. Such a conical singularity gives an additional contribution to the curvature and in effect the metric with a conical singularity ceases to be a saddle point of the action (1.1). There is however a single choice of e_0 for which the Euclidean time circle joins smoothly at $z = z_0$ and it is given by

$$e_0 = \frac{3}{8} N_c^2 \pi^2. \tag{1.24}$$

This calculation is an example of subtle interplay between consistency of gravitational description and properties of holographic matter. Similar reasonings are going to be used extensively throughout the Thesis.

Black objects are thermodynamic in nature [52] and in the context of AdS/CFT correspondence their thermodynamics is identified with thermodynamics of gauge theory plasma [51]. In particular, quarter of area density a of constant time sections of black brane event horizon (in Planck units) is identified with boundary entropy density[23]

$$s = \frac{a}{4 l_P^3} \tag{1.25}$$

and detailed calculation indeed shows an agreement with (1.21). Note also that the temperature T has an interpretation of Hawking temperature [53] of the black brane, which again confirms consistency of AdS/CFT approach.

Fefferman-Graham coordinates break down at the horizon (note that the determinant of metric vanishes at $z = z_0$) and it is useful to replace them with a better-adapted chart. An example of such can be given by ingoing Eddington-Finkelstein coordinates, in which black holes metric takes the form [10]

$$\mathrm{d}s^2 = 2\mathrm{d}\tilde{t}\mathrm{d}r - \frac{r^2}{\mathcal{L}^2}\left\{1 - \frac{(\pi \mathcal{L}^2 T)^4}{r^4}\right\}\mathrm{d}\tilde{t}^2 + \frac{r^2}{\mathcal{L}^2}\mathrm{d}\vec{x}^2. \tag{1.26}$$

Now boundary is located at $r = \infty$ and black hole curvature singularity at $r = 0$. As anticipated, the singularity is covered by the event horizon at $r = \pi\mathcal{L}^2 T$. In this new coordinate chart both the metric and its inverse are regular everywhere apart from $r = 0$[24]. Moreover, metrics (1.23) and (1.26) are related to each other by a *singular* coordinate transformation given by

[23] For planar black holes the total area of horizon is formally infinite.

[24] This is due to the absence of G_{rr} term in the metric.

$$
\begin{aligned}
r &= \frac{\mathcal{L}^2}{z}\sqrt{1+\frac{1}{4}\pi^4T^4z^4}, \\
\tilde{t} &= t+\frac{1}{4T}-\frac{1}{2\pi T}\arctan\left(\frac{r}{\pi\mathcal{L}^2T}\right)+\frac{1}{4\pi T}\log\frac{r-\pi\mathcal{L}^2T}{r+\pi\mathcal{L}^2T}.
\end{aligned} \tag{1.27}
$$

It is also worth stressing that ingoing radial null geodesics in ingoing Eddington-Finkelstein coordinates are curves of constant t, which means that an ingoing null signal is instantaneously transmitted into the bulk.

As anticipated, the *eternal* black brane (1.23)[25] corresponds to plasma in global equilibrium. The simplest dynamical situation is given by linearized perturbations of the bulk metric on top of the AdS-Schwarzschild black brane. Such perturbations must obey asymptotic AdS boundary conditions at $r = \infty$, but more importantly they have to fall into the horizon of the black brane[26]. Modes escaping from the horizon towards the boundary are not allowed by causal structure of spacetime, since the horizon acts as a surface of no return. This behavior of linearized 5-dimensional perturbations leads to complex dispersion relations for those modes (called quasinormal modes), which is a counterpart of dissipation in the boundary quantum field theory (see Chapter 2 for a discussion of those modes in the long-wavelength limit on quantum field theory side of correspondence).

[25] More precisely metric (1.23) covers small patch of full black brane Penrose diagram.
[26] Ingoing boundary conditions at black hole horizon.

Chapter 2

Conformal relativistic hydrodynamics

2.1 Modern relativistic hydrodynamics

Modern understanding of relativistic hydrodynamics is that of an effective field theory [12, 10]. Hydrodynamics describes long-distance (IR) near-equilibrium evolution of conserved quantities – the energy-momentum tensor and charge currents – and as such assumes local validity of thermodynamics. The relevant degrees of freedom are temperature[1] and fluid velocity, as well as densities of conserved charges if present in the system. Those are macroscopic quantities (IR observables) whose scales of changes (but not amplitudes of changes!) are required to be large compared to microscopic scale. When a quasiparticle picture is valid, the microscopic scale is set by the mean free path l_{mfp} and the hydrodynamic expansion parameter δ is given by

$$\delta = l_{mfp}/L, \tag{2.1}$$

where L denotes the characteristic scale of changes of relevant macroscopic quantities. In strongly coupled systems – examples of such are quark-gluon plasma at temperatures not much bigger than the transition temperature and holographic gauge theories at strong coupling – the microscopic scale is taken to be of order of temperature inverse on dimensional and physical grounds [12, 10]. In such cases hydrodynamic expansion parameter takes the form

$$\delta \sim \frac{1}{L\,T}. \tag{2.2}$$

The assumption of slow changes translates into the notion of gradient expansion around the locally equilibrated solution, namely the one containing no gradients. The effective field theory approach is based on including all irrelevant structures up to a desired order in gradient expansion of the energy-momentum tensor and conserved currents, as well as entropy current – hydrodynamic generalization of notion of entropy. This Thesis focuses on uncharged hydrodynamics and the only conserved macroscopic observable is the energy-momentum tensor (or more correctly its one-point function). The leading term in the expansion is provided by the energy-momentum tensor of equilibrated boosted plasma

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + (g^{\mu\nu} + u^\mu u^\nu)\, P + \dots \tag{2.3}$$

with both temperature T and velocity u^μ (which is normalized $u^\mu u_\mu = -1$) being functions of spacetime coordinates, whose behavior is specified by the equations of motion following from conservation of the energy-momentum tensor. The energy density ϵ and pressure P are

[1] or equivalently the energy density or pressure.

related by the equation of state $\epsilon = \epsilon(P)$. As anticipated before in Chapter 1, the equation of state in conformal case is dictated by tracelessness of the energy-momentum tensor and in 3+1 dimensions takes the form $\epsilon = 3P$ with both quantities scaling with temperature as T^4 on dimensional grounds.

Suppressed derivative (irrelevant) terms carry information about dissipation and relaxation processes in plasma and the perfect-fluid hydrodynamics given by (2.3) is not satisfactory. In particular, since in the comoving frame plasma locally "looks like" in equilibrium, the notion of entropy should, at least intuitively, make sense there. In the lab-frame there will be thus an entropy current flowing through the system, which in the leading approximation is just the product of thermodynamic entropy density and fluid velocity

$$J^{\mu} = s\, u^{\mu} \ldots \tag{2.4}$$

The second law of thermodynamics $\delta S \geq 0$ generalizes to an analogous statement about the divergence of the entropy current

$$\nabla_{\mu} J^{\mu} \geq 0. \tag{2.5}$$

It is a matter of direct calculation to check that the divergence of leading order entropy current vanishes on-shell, so that in perfect fluid hydrodynamics there is no dissipation (in a sense of entropy production). This means that suppressed quantities indeed carry new physics and both the energy-momentum tensor and entropy current have to be supplemented with additional contributions containing gradients of velocity and temperature, as well as metric of the manifold in which the quantum field theory lives[2].

The effective field theory approach to hydrodynamics is based on including *all* possible terms in gradient expansion of the energy-momentum tensor (and other conserved quantities if applicable) with decreasing relevance with the expansion terminated usually at first or second order. Such an approach was pioneered by Landau and Lifschitz in the case of first order hydrodynamics [54] and was revived recently in the context of second order hydrodynamics in [12, 10][3]. It has to be opposed to more phenomenological approach, where only certain (desired) second order terms were included [56, 57]. In particular, the perfect fluid energy-momentum (2.3) is to be supplemented with a dissipative part being symmetric tensor made of gradients of velocity, temperature and metric. Before defining the gradient terms entering the expansion it has to be specified what is meant by both temperature and velocity. One possible definition, so called Landau frame, states that the velocity of the fluid is the eigenvector of the energy-momentum tensor with the eigenvalue being $-\epsilon$

$$T^{\mu}_{\nu} u^{\nu} = -\epsilon\, u^{\mu}, \tag{2.6}$$

where ϵ is thermodynamic energy density, whose dependence on temperature is known[4]. Such definition is consistent with leading order expression (2.3) and implies that dissipative part of the energy-momentum tensor is orthogonal to velocity (i.e. it is a transverse tensor). Another important issue is that at the given order of the expansion not all gradient terms might be

[2]Note, that due to disspation, gradient expansion is performed at the level of equations of motion rather than action principle.

[3]Other important papers which found missing gradient terms in hydrodynamic expansion are [14] (for hydrodynamics with anomalous currents) and [55] (for magnetohydrodynamics). In all cases crucial insight or inspiration followed from string theory calculations.

[4]This definition implies that there is no momentum flow in local rest frame.

independent – some of them are usually equivalent on shell and only those which are not enter the gradient expansion. With these restrictions taken into account the most general energy-momentum tensor up to first order in derivatives takes the form [54]

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(g^{\mu\nu} + u^\mu u^\nu\right) P - \eta \sigma^{\mu\nu} - \zeta \Delta^{\mu\nu} \nabla_\alpha u^\alpha, \tag{2.7}$$

where

$$\begin{aligned} \sigma^{\mu\nu} &= \Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha, \\ \Delta^{\mu\nu} &= g^{\mu\nu} + u^\mu u^\nu \end{aligned} \tag{2.8}$$

and η and ζ are transport coefficients called shear and bulk viscosity respectively. Since each gradient produces additional power of energy (or inverse of L, which is identified with a macroscopic length scale of (2.1) and (2.2)), both η and ξ scale as T^3 on dimensional grounds. At the hydrodynamic level, differences between quantum field theories lie in concrete values of transport coefficients and the form of the equation of state[5], since microscopic degrees of freedom (e.g. quasiparticles in weakly coupled medium) are integrated out. Transport coefficients in the canonical approach[6] are functions of thermodynamic quantitites, as well as possibly dynamically generated scale (e.g. QCD). In order to compare transport properties of different (relativistic) quantum field theories one should focus on dimensionless intensive[7] quantities. For shear and bulk viscosities the relevant numbers are usually taken as their ratios to thermodynamic entropy density. In particular, the ratio of shear viscosity to entropy density in weakly coupled ($\lambda \ll 1$) gauge theories is very large [60, 61]

$$\frac{\eta}{s} \sim \frac{1}{\lambda^2 \log \lambda^{-1}}, \tag{2.9}$$

whereas for strongly coupled theories with classical two-derivative gravity dual lagrangian it takes the universal form [62, 63]

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{2.10}$$

in natural units. The result (2.10) played an important role in paradigm shift from weakly coupled gas of quarks and gluons to strongly coupled quark-gluon plasma at RHIC, since it suggests that small viscosity (in the sense of relevant ratio) might be a signal of strongly coupled regime. On the related note, the authors of [11] compared (2.10) with ratios of shear viscosity to entropy density of other systems and found out that it seems to be the lowest one[8]. This lead them to conjucture that actually the result (2.10) provides the lower bound on the relevant ratio, which stimulated a lot of interest in computing string theory (higher derivative) corrections to it (see Chapter 6 and [64], where very general set of higher order corrections to shear viscosity and other transport properties is provided, some of which are known to violate [19] the bound proposed in [11]).

[5]Note that the tensorial structure of the hydrodynamics is going to be different for conformal and non-conformal theories, but this can be taken into account by requiring that certain transport coefficients vanish.

[6]Not including any resummations, see however [58, 59]

[7]In a sense that they do not depend on the number of degrees of freedom in the leading order when this numer is taken to be large

[8]Currently the ratio shear viscosity to entropy density of strongly coupled quark-gluon plasma and fermions at unitarity seems to be of the same order of magnitude.

The energy-momentum tensor (2.7) is the most general one up to first order in gradients. However, the equations of motion it obeys are parabolic rather than hyperbolic and certain modes propagate with speeds exciting the speed of light. In order to restore causality, it is desirable to go to second order in gradient expansion, since this would make the equations of motion hyperbolic[9]. In Israel-Stewart framework [56, 57] the causality is restored by including single term of second order in gradients with corresponding transport coefficient called relaxation time[10]. This coefficient has been calculated for the first time at strong coupling in gravity dual to boost-invariant flow in [68] and using different hydrodynamic solution in [12]. The results disagreed and the authors of [12] proposed to resolve this inconsistency by including more terms in the hydrodynamic energy-momentum tensor at second order in gradients, so that the boost-invariant flow result was not really a relaxation time alone, but a sum of two different transport coefficients (see Section 3.3.1 and reference [12] for more details).

2.2 Conformal symmetry and allowed gradient terms

The effective field theory approach to hydrodynamics reviewed in this Chapter amounts to including all terms allowed by symmetries up to given order in gradients, in all applications in the literature so far it is at most second order (see however [58, 59]). For (holographic) conformal field theories, whose dynamics is a subject of this Thesis, the guiding symmetry principle highly constraining the form and number of possible gradient terms is conformal symmetry. Note that in conformal field theories in $3+1$ dimensions trace anomaly is made of squares of curvature tensor, so appears at fourth order in gradients in the hydrodynamic expansion[11]. This implies that the *hydrodynamic* energy-momentum tensor up to fourth order (so in particular at leading, first and second order in gradients) should be traceless and Weyl-covariant (see [12] for an excellent discussion on Weyl covariance in conformal hydrodynamics). The former requirement forces in particular bulk viscosity to be zero in *all* conformal field

[9]The causality violation happens for large wavelength modes, so beyond the validity of hydrodynamic description [65]. On the other hand, in numerical simulations such modes indeed propagate [66] and including second order terms is important. Second order hydrodynamics can be also understand as a theory improving the results of the first order approach (but not necessarily the regime of its validity – see Section 3.3.2 and references [67, 49] for a discussion on concrete example). In particular, second order effects influence estimates of shear viscosity and it is important to have a good control over their tensorial structure as well as intuition about the values of second order transport coefficients.

[10]The reasoning of Israel-Stewart approach can be summarized as follows. In the first order approach, the dissipative part of the energy-momentum tensor is given by the equation $T^{\mu\nu}_{dissipative} + \eta\sigma^{\mu\nu} + \zeta\Delta^{\mu\nu}\nabla_\alpha u^\alpha = 0$. This forumula means that $T^{\mu\nu}_{dissipative}$ relaxes instantaneously to its standard form. The causal theory should not have instantaneous phenoma and this drawback has been cured by Israel and Stewart by introducing relaxation time τ_Π^{IS} on the right hand side of the equation, so that it eventually reads in schematic form $T^{\mu\nu}_{dissipative} + \eta\sigma^{\mu\nu} + \zeta\Delta^{\mu\nu}\nabla_\alpha u^\alpha = \tau_\Pi^{\mathrm{IS}}\nabla T^{\mu\nu}_{dissipative}$. It has to be stressed that this approach is purely phenomenological and at the second order in gradient expansion there are more terms available, all of which need to be included for consistency of the description.

[11]Note that trace anomaly is $\langle T^\mu_\mu\rangle = \frac{c}{16\pi^2}\mathrm{I}_4 - \frac{a}{16\pi^2}\mathrm{E}_4$, where a and c are central charges of the theory and I_4 and E_4 are Euler density and square of Weyl curvature in 3+1 dimensions defined as [19]

$$\begin{aligned}
\mathrm{E}_4 &= R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \\
\mathrm{I}_4 &= R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2.
\end{aligned} \tag{2.11}$$

See also Chapter 6 for a discussion on coefficients in sitting in the trace anomaly in the context of higher derivative corrections to universal gravity action.

theories, while the latter implies that the hydrodynamic energy-momentum tensor at *low orders* of gradient expansion transform homogeneously (Weyl-covariantly) under the rescalings of the metric by an overall local factor (Weyl rescalings)

$$g_{\mu\nu} \to e^{-2\omega(x)}\, g_{\mu\nu}. \tag{2.12}$$

In the above formula the conformal weight of the metric is taken to be -2. In such a convention, the conformal weight of the hydrodynamic energy-momentum tensor in $3+1$ dimensions with both indices raised is 6, i.e.

$$T^{\mu\nu} \to e^{6\omega(x)}\, T_{\mu\nu}, \tag{2.13}$$

which can be derived from classical definition of the energy-momentum tensor. Moreover, the conservation equation of the energy-momentum tensor also transforms homogeneously in the orders of interest, as required by the self-consistency of this approach. This approximate symmetry of hydrodynamic equations implies that up to fourth order in gradients the hydrodynamic observables (the energy-momentum tensor, entrony current, conserved charges if present in a system) have to be written in terms of Weyl-covariant quantities, i.e. such that under Weyl rescalings transform homogenously (Weyl scalars, Weyl vectors and Weyl tensors).

The hydrodynamic degrees of freedom, T and u^μ, transform uniformly under Weyl transformations. In particular, velocity normalization condition $u_\mu u^\mu = -1$ makes the Weyl scaling of velocity transparent

$$u^\mu \to e^{\omega(x)} u^\mu, \tag{2.14}$$

whereas the conformal weight of temperature can be deduced from leading order expression for the energy-momentum tensor (2.3)

$$T \to e^{\omega(x)} T. \tag{2.15}$$

Note that if (and only if) entropy current is Weyl vector of weight 4, which is the case for leading order expression (2.4), its divergence also transforms homogeneously

$$\nabla_\mu J^\mu \to e^{4\,\omega(x)} \nabla_\mu J^\mu \tag{2.16}$$

and non-negativity property does not depend on Weyl rescalings in the orders of interest.

Weyl covariance allows for efficient construction of the energy-momentum tensor and entropy current of holographic conformal gauge theories in terms of elementary building blocks – Weyl-covariant transverse traceless tensors, transverse vectors and scalars containing given number of gradients. It is quite easy to understand why Weyl covariance is so restrictive. Since each gradient of Weyl rescaled velocity, temperature or metric produces derivative of Weyl factor $\omega(x)$, which should not appear up to the order when the conformal anomaly enters, to obey the symmetry of Weyl covariance gradients must be combined in such a way, so that all derivatives of $\omega(x)$ cancel out and the sum of all terms transforms homogeneously under Weyl rescalings. Using the notation of [13] (which descends from [69]) one has at second order[12] 5 conformal

[12] In the hydrodynamic formulas $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is projector to the fluid's local rest frame, whereas $\nabla^\mu_\perp = \Delta^{\mu\nu}\nabla_\nu$. Moreover fluid shear tensor (responsible for dissipation in the first order conformal hydrodynamics) reads $\sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) - \frac{1}{4-1}\Delta^{\mu\nu}\nabla_\alpha u^\alpha$ and vorticity (nonzero for rotating fluid) $\Omega^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha\right)$.

tensors

$$\begin{aligned}
\mathcal{O}_1^{\mu\nu} &= R^{<\mu\nu>} - c_s^2 \left(2\nabla_\perp^{<\mu}\nabla_\perp^{\nu>}\ln s + \sigma^{\mu\nu}\left(\nabla\cdot u\right) - 2c_s^2\nabla_\perp^{<\mu}\ln s\nabla_\perp^{\nu>}\ln s\right) ,\\
\mathcal{O}_2^{\mu\nu} &= R^{<\mu\nu>} - 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} ,\\
\mathcal{O}_3^{\mu\nu} &= \sigma^{<\mu}{}_\lambda\sigma^{\nu>\lambda} , \quad \mathcal{O}_4^{\mu\nu} = \sigma^{<\mu}{}_\lambda\Omega^{\nu>\lambda} , \quad \mathcal{O}_5^{\mu\nu} = \Omega^{<\mu}{}_\lambda\Omega^{\nu>\lambda} ,
\end{aligned} \tag{2.17}$$

3 possible conformal (Weyl-covariant) scalars

$$\mathcal{S}_1 = \sigma_{\mu\nu}\sigma^{\mu\nu} , \quad \mathcal{S}_2 = \Omega_{\mu\nu}\Omega^{\mu\nu} ,$$
$$\mathcal{S}_3 = c_s^2\nabla_\mu^\perp\nabla_\perp^\mu \ln s + \tfrac{c_s^4}{2}\nabla_\mu^\perp\ln s\nabla_\perp^\mu\ln s - \tfrac{1}{2}u_\alpha u_\beta R^{\alpha\beta} - \tfrac{1}{4}R + \tfrac{1}{6}\left(\nabla\cdot u\right)^2 \tag{2.18}$$

and 2 possible conformal vectors[13]

$$\mathcal{V}_1^\mu = \nabla_\alpha^\perp\sigma^{\alpha\mu} + 2c_s^2\sigma^{\alpha\mu}\nabla_\alpha^\perp\ln s - \frac{u^\mu}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} , \quad \mathcal{V}_2^\mu = \nabla_\alpha^\perp\Omega^{\mu\alpha} + u^\mu\Omega_{\alpha\beta}\Omega^{\alpha\beta} . \tag{2.19}$$

Note that at first order the only conformal quantity is the shear tensor. This implies that conformal field theories[14] are characterized by single transport coefficient at first order and five others at second order in gradients[15] with the most general energy-momentum tensor in the absence of conserved charges reading [10, 12, 13]

$$\begin{aligned}
T^{\mu\nu} &= \epsilon u^\mu u^\nu + P\,\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \\
&\quad \eta\,\tau_\Pi\left\{\mathcal{O}_1^{\mu\nu} - \mathcal{O}_2^{\mu\nu} - \frac{1}{2}\mathcal{O}_3^{\mu\nu} - 2\mathcal{O}_5^{\mu\nu}\right\} + \kappa\mathcal{O}_2^{\mu\nu} + \lambda_1\mathcal{O}_3^{\mu\nu} + \lambda_2\mathcal{O}_4^{\mu\nu} + \lambda_3\mathcal{O}_5^{\mu\nu} .
\end{aligned} \tag{2.20}$$

τ_Π is an analog of Israel-Stewart relaxation time and κ, $\lambda_{1,2,3}$ are other transport coefficients of conformal fluids. In particular, for vorticity-free ($\Omega_{\mu\nu} = 0$) flows in flat spacetimes only two tensorial structures contribute to the energy-momentum tensor at second order, so that the flow is sensitive only to values of τ_Π and λ_1 besides shear viscosity η. An example of such solution is boost-invariant hydrodynamics discussed extensively in the next Chapter.

Equations of hydrodynamics support linearized perturbations – shear and sound waves, which are respectively transverse and longitudinal modes (relative to direction of propagation) . Out of all transport coefficients appearing in the energy-momentum tensor up to second order in derivatives, linearized fluctuations in flat background are sensitive only to shear viscosity η and relaxation time τ_Π[16]. Sound waves propagating in direction x^3 are perturbations of energy density ϵ, pressure P (those perturbations are related to each other by the equation of state) and 2 components of velocity u^0 and u^3 (related to each other by normalization condition $u_\mu u^\mu = -1$), whose spacetime dependence is harmonic and reads $\exp\left(-i\,\omega\left(k\right)x^0 + i\,k\,x^3\right)$. The dispersion relation $\omega\left(k\right)$ for large wavelengths (compared to temperature – hydrodynamic limit) takes the form [12]

$$\omega\left(k\right) = \pm c_s\,k - i\Gamma k^2 \pm \frac{\Gamma}{c_s}\left(c_s^2\tau_\Pi - \frac{\Gamma}{2}\right)k^3 + \mathcal{O}\left(k^4\right) \tag{2.21}$$

with speed of sound being $c_s = \frac{1}{\sqrt{3}}$[17] and sound attenuation $\Gamma = \frac{2}{3}\frac{\eta}{\epsilon+P}$. Once the form of sound-

[13]Note the absence of parity breaking terms present in [69]. For discussion of hydrodynamics with parity breaking terms see [14].

[14]In dimensions 1+2 and higher, since in 1+1 dimension the energy-momentum tensor is trivial, see [70]

[15]In non-conformal case there are 15 independent tensorial structures apearing at second order in derivatives [13].

[16]All tensorial structures up to second order in gradients apart from $\sigma^{\mu\nu}$ and combination $\mathcal{O}_1^{\mu\nu} - \mathcal{O}_2^{\mu\nu} - \frac{1}{2}\mathcal{O}_3^{\mu\nu} - 2\mathcal{O}_5^{\mu\nu}$ are nonlinear in gradients.

[17]≈ 0.58 – propagation of sound waves in relativistic hydrodynamics is a relativistic process

wave dispersion relation is known, it allows for efficient calculation of shear viscosity η and relaxation time τ_{Π}. AdS/CFT correspondence maps sound waves in holographic gauge theories to linearized gravitational perturbations on top of AdS Schwarzschild black hole of respective polarization with ingoing boundary conditions at the horizon specifying the coefficients in the dispersion relation, thus some of transport properties of holographic gauge theories [71, 72, 73][18].

2.3 Entropy current and hydrodynamics

The requirement that entropy should be non-decreasing during hydrodynamic evolution can be expressed in a covariant way in terms of an entropy current whose divergence is non-negative (2.5). While the energy-momentum tensor is a canonically defined operator, the entropy current is a derived notion. In the spirit of hydrodynamics (or effective field theory) it is also constructed in a gradient expansion as a sum of all possible terms at a given order. The dynamical equations of hydrodynamics are the conservation equations for the expectation value of the energy-momentum tensor. Thus, the coefficients appearing in the gradient expansion of the expectation value of the energy-momentum tensor (the transport coefficients) are the physical parameters of this phenomenological theory, since they figure directly in the evolution equations. They describe physical properties of the underlying quantum field theory. In contrast, the coefficients which appear in the phenomenological expression for the entropy current are constrained only by the requirement that its divergence be non-negative. These parameters are logically independent of the transport coefficients. At the present level of understanding they reflect a real ambiguity in the phenomenological notion of entropy current in hydrodynamics (as explained in the rest of the section). This ambiguity is however of no consequence when entropy differences between equilibrium states are considered.

In the case of conformal fluids the most general form of the entropy current was recently constructed [69, 13] up to second order in gradients

$$\begin{aligned} S^{\mu}_{\mathrm{non-eq}} &= s u^{\mu} + \frac{A_1}{4}\mathcal{S}_1 u^{\mu} + A_2 \mathcal{S}_2 u^{\mu} + A_3\left(4\mathcal{S}_3 - \frac{1}{2}\mathcal{S}_1 + 2\mathcal{S}_2\right)u^{\mu} \\ &\quad + B_1\left(\frac{1}{2}\mathcal{V}^{\mu}_1 + \frac{u^{\mu}}{4}\mathcal{S}_1\right) + B_2\left(\mathcal{V}^{\mu}_2 - u^{\mu}\mathcal{S}_2\right). \end{aligned} \tag{2.23}$$

Here s denotes the thermodynamic entropy density (3.9), and $\mathcal{S}_{1,2,3}$ are conformal scalars and $\mathcal{V}_{1,2}$ are conformal vectors.

[18]The analogous statement holds for shear waves, which are perturbations of ϵ, P, u^0 and u^1 (later two are related to each other through normalization condition) with the same spacetime dependence and dispersion relation reading

$$\omega\left(k\right) = -ihk^2 - ih^2\tau_{\Pi}k^4 + O\left(k^6\right), \tag{2.22}$$

where $h = \frac{\eta}{\epsilon+P}$. Note that two dispersion relation are different, yet they should yield the same transport coefficients. This is very non-trivial check of correctness of transport coefficients obtained holographically.

The entropy current depends (2.23) on 5 constants $A_{1,2,3}$ and $B_{1,2}$ and its divergence reads

$$\begin{aligned}\nabla_\mu S^\mu_{\mathrm{non-eq}} \;=\;& \frac{1}{2}\nabla^\perp_\mu \nabla^\perp_\nu \sigma^{\mu\nu}\left(-2A_3+B_1\right)+\frac{1}{3}\nabla^\perp_\mu\sigma^{\mu\nu}\nabla^\perp_\nu \ln s\left(-2A_3+B_1\right)\\ &+\sigma_{\mu\nu}\left[\frac{\eta}{2T}\sigma^{\mu\nu}+R^{\mu\nu}\left(-\frac{\kappa}{2T}+A_3\right)+u_\alpha u_\beta R^{\alpha<\mu\nu>\beta}\left(\frac{\kappa-\eta\tau_\pi}{T}+A_1+B_1-2A_3\right)\right.\\ &-\frac{1}{4}\sigma^\mu{}_\lambda\sigma^{\nu\lambda}\left(\frac{2\lambda_1-\eta\tau_\pi}{T}+A_1+B_1-2A_3\right)+\frac{1}{3}\nabla^{<\mu}_\perp\nabla^{\nu>}_\perp \ln s\left(\frac{\eta\tau_\pi}{T}-A_1-2A_3\right)\\ &+\Omega^\mu{}_\alpha\Omega^{\nu\alpha}\left(-\frac{\lambda_3+2\eta\tau_\pi}{2T}+A_1-2A_2-2A_3+B_1\right)\\ &\left.+\sigma^{\mu\nu}\frac{(\nabla\cdot u)}{12}\left(\frac{2\eta\tau_\pi}{T}-2A_1+6A_3-5B_1\right)+\frac{1}{9}\nabla^{<\mu}_\perp \ln s\nabla^{\nu>}_\perp \ln s\left(-\frac{\eta\tau_\pi}{T}+A_1+B_1\right)\right].\end{aligned} \tag{2.24}$$

If the shear tensor is non-vanishing, which is a generic situation[19], the positivity of the shear viscosity η should guarantee (see however [13] and the next footnote) that divergence of the entropy current is non-negative: higher order terms cannot change this conclusion as long as the gradient expansion is valid. However, as noted in [69], it is perfectly reasonable for $\sigma^{\mu\nu}$ to locally vanish (requiring this imposes just 5 conditions for derivatives of the four-velocity) in which case the higher order terms will dominate the entropy production. Positivity of (2.24) thus requires

$$B_1 = 2\,A_3. \tag{2.25}$$

Since the shear tensor $\sigma^{\mu\nu}$ is multiplying the whole square bracket in (2.24), in the case when it is 0 the whole contribution from first two orders is absent. At this level there is a real 4-parameter ambiguity in the hydrodynamic construction of the entropy current[20].

2.4 Fluid/gravity duality

Hydrodynamic modes are understood as long-wavelength perturbations on top of locally equilibrated plasma and their leading order energy-momentum tensor matches precisely the form of the one of equilibrated boosted plasma with boost parameter u^μ and temperature T understood as slowly varying functions of position. On the other hand, the gravity dual to uniformly boosted plasma is given by boosted AdS-Schwarzschild black hole

$$\mathrm{d}s^2 = 2u_\mu \mathrm{d}x^\mu \mathrm{d}r - \frac{r^2}{\mathcal{L}^2}\left\{1-\frac{(\pi\mathcal{L}^2T)^4}{r^4}\right\}u_\mu u_\nu \mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{r^2}{\mathcal{L}^2}(\eta_{\mu\nu}+u_\mu u_\mu)\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{2.26}$$

The idea behind fluid/gravity duality [10] was to promote T and u^μ from formula (2.26) to be slowly varying (in hydrodynamic sense) functions of position and treat (2.26) as an approximate solution of Einstein equations valid in the leading order in gradients. The metric (2.26) gets corrected by gradient terms, which give rise to first and second order hydrodynamics in the boundary quantum field theory. An important feature of the construction [10] is that both the leading order metric (2.26) as well as subleading corrections are regular all the way to black brane singularity at $r=0$ with singularity covered by the event horizon [69]. The regularity of bulk metric is attributed to well-adapted coordinate choice being ingoing

[19] Note that $\sigma_{\mu\nu}\sigma^{\mu\nu}$ as a trace of the square of matrix cannot be negative and vanishes if and only if $\sigma^{\mu\nu}=0$.

[20] Considerations of the case with an arbitrary small $\sigma^{\mu\nu}$ in [13] suggest that further constraints on the entropy current may be imposed. In particular, the only freedom left after such consideration is in the parameter A_1. It appears that these arguments rest on competition between terms of different orders in the gradient expansion.

Eddington-Finkelstein ones and this choice is also used in Chapters 4 and 5 to show that gravity dual to boost-invariant hydrodynamics is indeed a regular spacetime.

Chapter 3

Boost-invariant flow

3.1 Toy-models of plasma dynamics

The main motivation to do RHIC-inspired field theory calculations using gravitational prescription is to understand thermalization time, initial conditions for hydrodynamic evolution and transport properties of strongly coupled mediums resembling real-world quark-gluon plasma[1]. There has been an enormous progress in obtaining the latter quantities and that part of applications of gauge/gravity duality is rather well explored[2]. In particular, all phenomena involving hydrodynamic evolution of holographic matter are captured within the framework of fluid/gravity duality [10]. Once transport properties of the theory in question are known there is usually no point in studying the gravity dual picture[3] and currently far-from-equilibrium applications of holographic methods seems to be the most exciting ones. However, one has to bear in mind that some parts of early time dynamics of QCD plasma might be governed by weak coupling and the results obtained for holographic gauge theories might not be directly applicable to QCD even at crude qualitative level. It still does not mean that AdS/CFT methods are of no relevance for nonequilibrium QCD. They are currently the only tools to understand thermalization process in strongly coupled gauge theories and their experimental relevance depends on the existence of pre-equilibrium strongly coupled phase. Very short thermalization time of QCD matter at RHIC suggests that this might be indeed the case. On a related note, it should be stressed here that hadronization of QCD matter is much beyond the scope of non-conformal gravitational methods. This is because the applicability of classical gravitational description requires the large-N_c limit on the field theory side in which the plasma-like configurations are dominant and hadronization is suppressed by a factor of N_c^{-2}[4]. Intuitively this means that holographic expanding mediums are going to cool down indefinitely[5].

[1] There was also a considerable interest in the behavior of hard probes in the holographic plasmas, but this topic is beyond the scope of the Thesis.

[2] Concrete values of transport coefficients of holographic gauge theories beyond the universal gravity action are model-dependent, e.g. they differ for theories with or without SUSY (see references [74, 36, 74, 75, 64, 76] and Chapter 6 for more details).

[3] Fluid/gravity duality might also serve as a tool to understand the properties of black hole space times using the equations of hydrodynamics [77].

[4] This means that confinement/deconfinement phase transition is of the first order. For real-world QCD it is a crossover.

[5] Confront this with gravity dual to cascading gauge theory [78, 79, 80], which is perturbatively unstable below certain temperature lower than the critical temperature [81]. In the expanding plasma scenario (e.g. boost-invariant flow), this temperature is going to be achieved in finite time possibly triggering the instability (the author thanks Ofer Aharony for pointing this out and for various discussions on that point). The fate of this instability is unknown. It would be very interesting to understand the nonlinear dynamics driving such

Taking the limitations of applied AdS/CFT framework into account, this Thesis develops gravitational methods for studying nonlinear nonequilibrium physics of strongly coupled holographic quantum field theories. For simplicity, only conformal examples of gauge/gravity duality are considered. Ideally, hCFT dynamics of interest should resemble, at least at the superficial level, the one of QCD matter at RHIC. In the first, very crude, approximation one can think of the central collision of nuclei as a process involving collision of two pancakes made of gauge theory matter, which fill all the transverse space and at same time are infinitely thin. The collision process should create debris, which eventually will become an expanding "quark-gluon plasma", and two receding "wounded" projectiles.

Single projectile can be toy-modelled by the field theory configuration with non-vanishing $--$ component of the expectation value of the energy-momentum tensor [40][6]

$$T_{--} = \mu\delta(x^-). \tag{3.1}$$

The AdS/CFT dual of a single projectile is just a shock-wave metric[7]

$$\mathrm{d}s^2 = \frac{-\mathrm{d}x^-\mathrm{d}x^+ + \mu\, z^4\, \delta\,(x^-)\; \mathrm{d}x^{-\,2} + \mathrm{d}\mathbf{x}^2}{z^2} + \frac{\mathrm{d}z^2}{z^2}, \tag{3.2}$$

which is an exact solution of Einstein equations for any lab-frame time t. The expectation value of the energy-momentum tensor before the collision is given by the sum of the energy-momentum tensors of the projectiles and the gravity dual is thus a superposition of two shock-waves. Such metric is a valid solution of equations of motion for $t < 0$ and the subsequent dynamics has to be determined by solving Einstein equations with superposed metric taken as an initial condition at some $t_{\mathrm{ini}} < 0$. It should be noted that there is no clear physical reason for the energy-momentum tensor on the light-cone to be the same or similar before and after the collision. This means that one should not regard the evolution after the collision as a small perturbation on top of the two, almost unchanged, shock-waves. In particular, such calculations lack sensible dimensionless perturbative parameters and presumably the only reliable approach to solve the problem of shock-wave collision requires using numerical methods from the beginning. Postponing this issue for possible future work, this Thesis focuses on another toy-model, which captures some of the physics of interest (namely nontrivial far-from-equilibrium dynamics involving thermalization with subsequent hydrodynamic evolution) and at the same time leads to simpler gravitational dual. This example of field theory dynamics is provided by Bjorken's boost-invariant flow [33].

3.2 General features of boost-invariant dynamics

Boost-invariant flow of interest is a one-dimensional expansion of plasma with the boost symmetry along the expansion axis as well as rotational and translational symmetry in the perpendicular plane[8]. The assumption of boost-invariance is more transparent in the coordinate system which makes it manifest. If x^0 denotes the lab frame time and x^1 is the cartesian

instability in an expanding setup. It is also worth stressing that in general, non-conformal plasmas exhibit richer dynamics, e.g. bulk viscosity peaking near the phase transition might lead to plasma cavitation for certain initial conditions [82].

[6] Note that in hCFTs the projectiles of interests consist of N_c^2 degrees of freedom, so are "plasma-like" states.

[7] For an extensive discussion of shock-waves metrics in this context see [83].

[8] In literature the latter assumptions are often lifted, leading to more realistic dynamics, see e.g. [84].

coordinate along the expansion axis, then the transformation to more convenient coordinates – proper time τ and rapidity y – takes the form

$$x^0 = \tau \cosh y \quad \text{and} \quad x^1 = \tau \sinh y. \tag{3.3}$$

Note that proper time is invariant under boosts along the expansion axis, whereas rapidity is not (it is shifted by a constant). This means that the assumption of boost-invariance in proper time - rapidity coordinates translates into the requirement that physical observables do not depend on rapidity. Boost-invariant flow is thus an example of nontrivial $(1+1)$-dimensional dynamics, which effectively depends on a single evolution parameter, the proper time τ. It is worth noting that proper time dependence leads to non-vanishing gradients with respect to the space-like coordinate x^1 and thus allows for hydrodynamic behavior. This would not be the case if the field theory dynamics depended only on lab-frame time. It is also worth stressing that proper time and rapidity are curvilinear coordinates in which the Minkowski metric takes the nontrivial form

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tau^2 \mathrm{d}y^2 + \mathrm{d}\mathbf{x}^2 \,. \tag{3.4}$$

The metric in this coordinate frame has a couple of interesting properties, which will be important for later applications. Firstly, as expected on general grounds, it does not depend on rapidity. Moreover, the curvilinear character of the coordinate system leads to nonvanishing Christoffel symbols, as well as nontrivial form of the constant proper time volume element, which increases linearly with proper time. Lastly, the point $\tau = 0$, corresponding to the boundary of the light-cone in x^0 and x^1, is regular in analogy with cylindric or spherical coordinates in euclidean signature. In particular, it makes perfect sense to expand physical quantities around $\tau = 0$, as well as $\tau = \infty$ and it is natural to expect finite results.

This Thesis studies the universal sector of decoupled dynamics of the energy-momentum tensor of holographic conformal field theories with the use of gravitational techniques. The most general one, which obeys the symmetries of the problem[9] and is conserved and traceless[10] is fully expressible in terms of a single function which is the energy density $\epsilon(\tau)$ [40]

$$T_{\mu\nu} = \operatorname{diag}\left\{\epsilon(\tau)\,,\, -\tau^3\epsilon'(\tau) - \tau^2\epsilon(\tau)\,,\, \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau)\,,\, \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau)\right\}\,. \tag{3.5}$$

All the physics of interest is hidden within a single unknown function – the energy density $\epsilon(\tau)$[11].

The main physical questions are to understand the behavior of holographic gauge theory plasma at late (near-equilibrium physics) and early time (far-from-equilibrium physics). At late time, one expects hydrodynamic behavior and would like to derive its properties – low viscosity, transport coefficients, etc. At early time, one would like to understand the rapid thermalization starting from ultra-relativistic initial conditions, with transient times linking the two regimes.

[9] And is also invariant under parity transformations in rapidity. For studies of the Bjorken flow with this assumption lifted see [85].

[10] Note that these requirements comes naturally from solving Einstein equations in the near-boundary expansion.

[11] Overall τ^2 in T_{yy} component of the energy-momentum tensor comes from the metric with lower indices and does not lead to any pathologies.

3.3 Boost-invariant flow near equilibrium

3.3.1 Bjorken hydrodynamics

It is natural to expect that at late time the plasma is locally equilibrated (thus is isotropic). This would imply, that in the leading approximation longitudinal $p_{\|} = T^y_y$ and transverse pressures $p_\perp = T^x_x$ are equal leading to the differential equation for the energy density

$$- \epsilon(\tau) - \tau\epsilon'(\tau) = \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau) \,. \tag{3.6}$$

This equation has a simple solution

$$\epsilon(\tau) \sim \frac{1}{\tau^{4/3}} \,, \tag{3.7}$$

which after taking into account that $\epsilon \sim T^4$ in local equilibrium[12] leads to

$$T \sim \frac{1}{\tau^{1/3}} \,. \tag{3.10}$$

The result is exact in the large-τ limit, but otherwise will get corrected. These corrections will be important, since they are responsible for entropy production. In particular, note that although the entropy density $s \sim T^3$ decreases with proper time, total entropy stays constant because of nontrivial volume element which scales linearly with proper time.

There will be two types of corrections to the energy density (3.7)[13], which are associated with dissipative processes (entropy production). Because proper time dependence introduces nontrivial spatial gradients, the leading corrections will come from hydrodynamic gradient expansion. Those will arise as power-like tails. There will be also exponential corrections dying off very quickly at late time, but otherwise responsible for thermalization processes.

Following the hydrodynamic prescription, each gradient of the physical quantity is suppressed by the inverse of the temperature[14]. Since gradients produce additional powers of $1/\tau$ and temperature scales asymptotically as $T \sim 1/\tau^{1/3}$, one should expect hydrodynamic expansion to be in integer powers of $1/\tau^{2/3}$. The equations of second order hydrodynamics with T being the function of proper time only and boost-invariant velocity specified by

$$u^\mu = [\partial_\tau]^\mu \tag{3.11}$$

[12] Since for a conformal fluid the equation of state is $p = 1/3\epsilon$, one has

$$\begin{aligned} \epsilon &= e_0 T^4 \,, & (3.8) \\ s &= \frac{4}{3} e_0 T^3 \,, & (3.9) \end{aligned}$$

where s is the usual thermodynamic entropy density.

[13] Note that the notion of the temperature should make sense only in local equilibrium. The energy density is a general concept and requires no assumptions.

[14] Or equivalently the hydrodynamics expansion parameter is given by $\frac{1}{L\,T}$, where L is the scale associated with gradients of macroscopic quantities and $\frac{1}{T}$ is the microscopic scale, which is integrated out.

take the form (see [12])

$$\begin{aligned}\partial_\tau\epsilon &= -\frac{4}{3}\frac{\epsilon}{\tau}+\frac{\Phi}{\tau}\,,\\ \tau_\Pi\partial_\tau\Phi &= \frac{4}{3}\frac{\eta}{\tau}-\Phi-\frac{4}{3}\frac{\tau_\Pi}{\tau}\Phi-\frac{1}{2}\frac{\lambda_1}{\eta^2}\Phi^2,\end{aligned} \tag{3.12}$$

where Φ is the yy component of the shear tensor $\Pi^{\mu\nu}$ (dissipative part of the energy-momentum tensor). These equations determine the proper time evolution of the energy density, which in turn specifies the dependence of temperature on τ. Substituting the form of the energy density (together with the equation of state) into these equations leads to the following, confirming previous intuition, perturbative solution for the temperature as a function of proper time

$$T\left(\tau\right)=\frac{\Lambda}{\tau^{1/3}}\left\{1-\frac{1}{\Lambda\,\tau^{2/3}}\cdot\frac{\eta_0}{2}+\frac{1}{\Lambda^2\,\tau^{4/3}}\cdot\left(\frac{\lambda_1^{(0)}}{6}-\frac{\eta_0\tau_\Pi^{(0)}}{6}\right)+\ldots\right\}, \tag{3.13}$$

where Λ is a scale fixed by the initial conditions and the only arbitrary number in the construction[15]. In much of the literature (such as [40, 86, 68, 16]) the choice $\Lambda=\frac{\sqrt{2}}{3^{1/4}\pi}$ is made. The constants η_0, $\tau_\Pi^{(0)}$, $\lambda_1^{(0)}$ are various transport coefficients from the first (η_0) and second order viscous hydrodynamics[16]. They are universal numbers related to the microscopic physics of the underlying quantum field theory. Their presence signals the dissipative nature of the flow – the entropy production.

The constant Λ is a dimensionful quantity, which sets the overall scaling of temperature with proper time. The formula (3.13) makes it transparent why hydrodynamics is universal – no matter what were the (boost-invariant) initial conditions in the past, at late time the only quantity which cares about them is a single dimensionful number Λ. Note that due to symmetries and flat background, at second order only two out of five allowed gradient terms enter. Since relaxation time can be obtained from linearized hydrodynamics, the boost-invariant flow allows for *efficient* calculation of the coefficient λ_1 in different models.

3.3.2 Validity of the hydrodynamic description

The modern view of hydrodynamics is similar to that of effective field theory. It is a phenomenological description of phenomena on scales much larger than those of their microscopic dynamics, constructed as a systematic expansion in gradients. In the context of boost-invariant flow this translates into an expansion in powers of $1/\tau^{2/3}$. As in the case of any perturbative expansion, one needs to observe the regime where the expansion can be reasonably expected to apply. A criterion for this is that the subleading terms in the expansion be smaller than the leading order. In the context of Bjorken flow this can be understood as a condition on the

[15]It is easy to understand the presence of Λ if one considers the dilatation $x^\mu\to\alpha\,x^\mu$. Then $\tau\to\alpha\,\tau$ and $\Lambda\to\alpha^{-2/3}\Lambda$ which leads to $\epsilon\left(\alpha\tau\right)=\alpha^{-4}\epsilon\left(\tau\right)$ being the correct scaling.

[16]More precisely these are dimensionless numbers related to the transport coefficients by the relations

$$\begin{aligned}\eta &= \eta_0 e_0 T^3,\\ \tau_\Pi &= \tau_\Pi^0 T^{-1},\\ \lambda_1 &= \lambda_1^0 e_0 T^2,\end{aligned} \tag{3.14}$$

where $e0$ is defined as previously by $\epsilon=e_0T^4$.

minimal time when the expansion can be trusted. For the first subleading term in (3.13) to be smaller than the leading order[17] one needs $\tau > \tau_{min}$, where

$$\frac{1}{\Lambda\, \tau_{min}^{2/3}} \cdot \frac{\eta_0}{2} \equiv \alpha < 1\,. \tag{3.15}$$

One requires that the expansion of any physical quantity such as energy or entropy density should have this property.

Note that including higher order terms does not extend the regime of validity of the hydrodynamic expansion, but rather improves the accuracy within the hydrodynamic window. Moreover, in a general boost-invariant dynamical situation τ_{min} does not coincide with thermalization time, since the non-hydrodynamic (exponential) modes do not have to be negligible at that time [67].

3.3.3 Boost-invariant entropy current

From the perspective of the AdS/CFT correspondence it is natural to ask whether the ambiguities appearing in the construction of the hydrodynamic entropy current match on both sides of the duality. In a general situation this might be involved, but one may try to gain some insight into this question by considering a particular solution. The Bjorken flow provides a simple, highly symmetric, yet nontrivial example.

The current (2.23) evaluated on the boost-invariant solution given by the velocity $u^\mu = [\partial_\tau]^\mu$ and temperature $T(\tau)$ (3.13) takes the form

$$J^\mu = \tilde{s} u^\mu \tag{3.16}$$

with

$$\tilde{s}(\tau) = s(T(\tau)) \left\{ 1 + 2\, \frac{A_1 - A_3 + B_1}{3\pi^2 T(\tau)^2 \tau^2} \right\}. \tag{3.17}$$

In general the entropy current does not have to be proportional to the flow velocity beyond leading order (perfect fluid), but in the special case of boost-invariant flow non-leading order effects are captured by the single scalar function $\tilde{s}(\tau)$. This function involves 3 of the 5 constants appearing in the general phenomenological construction. Changing the value of A_1 has been identified with the freedom in choosing the horizon to boundary map [69]. The ambiguity parametrized by A_3 was not interpreted in [69]. One would like to interpret this freedom in terms of allowed definitions of “horizon” on gravity side. Note that the example of Bjorken flow, while rather special, is still rich enough to partially capture this ambiguity.

In quantitative terms this ambiguity can be estimated as follows. In order for the hydrodynamic expansion of (3.17) to be valid the magnitude of $|A_1 + A_3|$ should bounded so that the leading term dominates for times larger than τ_{min} defined by (3.15). Expanding (3.17) up to second order one has

$$\tilde{s}(\tau) \sim \frac{\Lambda^3}{\tau} \left\{ 1 - \frac{1}{2\pi\Lambda} \frac{1}{\tau^{2/3}} + \frac{(8(A_1 + A_3) + \log(2))}{12\pi^2\Lambda^2} \frac{1}{\tau^{4/3}} \right\}. \tag{3.18}$$

Demanding that the second order contribution be smaller than the first order correction by a factor of $\alpha\beta$ at $\tau = \tau_{min}$ leads to the bound

$$\frac{1}{8}(-\beta - \log(2)) < A_1 + A_3 < \frac{1}{8}(\beta - \log(2))\,, \tag{3.19}$$

[17] The temperature has been chosen here because it enters the definition of the gradient expansion.

where β is at most of order $1/\alpha$. This provides a rough estimate of the allowed indeterminacy in the phenomenological notion of non-equilibrium entropy as defined by (3.17)

$$\tilde{s}(\tau) \sim \frac{\Lambda^3}{\tau} \left\{ 1 - \frac{1}{2\pi\Lambda} \frac{1}{\tau^{2/3}} \pm \frac{\beta}{12\pi^2\Lambda^2} \frac{1}{\tau^{4/3}} \right\} . \tag{3.20}$$

One would like to understand this quantitatively in terms of the freedom of defining entropy on the gravity side. This has been done in [49] and is a subject of Chapter 5.

Chapter 4

Near-equilibrium dynamics of the boost-invariant flow from supergravity

4.1 Bulk non-singularity condition and boost-invariant hydrodynamics

AdS/CFT correspondence relates physical configurations of gauge fields at large N_c and strong coupling to naked singularity-free dual geometries. As anticipated before, checking regularity of geometry requires solving Einstein's equations beyond the near-boundary power series (1.10) and in the general case is expected to involve advanced numerical methods. A notable exception is the boost-invariant holographic gauge theory dynamics, in which case Einstein's equations can be solved exactly in the radial direction and approximately in proper time. The regimes of special interest are late (where near-equilibrium dynamics is expected) and early proper time (regime of far-from-equilibrium evolution)[1] linked by an approach to local equilibrium at transient time.

In the pioneering work [40] Janik and Peschanski showed using only mild assumptions and working in Fefferman-Graham coordinates that non-singularity of the dual gravity description forces the boost-invariant solution at late time to be governed by perfect fluid hydrodynamics (see next Section for a brief introduction). Subsequent developments included solving Einstein's equations in the late-time expansion being, as now understood, bulk counterpart of boundary gradient expansion and calculating shear viscosity [86][2] and Israel-Stewart [56, 57] relaxation time [68][3], all of which was completed prior to formulation of fluid/gravity duality [10][4]. The key requirement, which fixed integration constants related to transport coefficients of holographic gauge theory undergoing boost-invariant evolution, was non-singularity of the

[1] It turns out that in Fefferman-Graham coordinates gravitational constraints can be easily solved only at $\tau_{FG} = 0$. Contrary, in Eddington-Finkelstein coordinates there is no major benefit from considering $\tau_{EF} = 0$ and constraints can be solved for any τ_{EF}. Note also that surfaces $\tau_{FG} = \text{const}$ and $\tau_{EF} = \text{const}$ differ in the bulk despite coinciding at the boundary.

[2] Yielding standard result $\eta/s = 1/4\pi$, which at the time of publication [86] was regarded as a non-trivial cross-check of the whole framework introduced in [40].

[3] Israel-Stewart relaxation time calculated in [68] is actually linear combination of relaxation time τ_Π and λ_1 – see extensive discussion in previous Chapter.

[4] Actually gravity dual to boost-invariant flow at late time should be regarded as a special example of fluid/gravity duality. The difference between Janik and Peschanski approach and this of fluid/gravity duality lies in a different choice of coordinate chart being Fefferman-Graham coordinate frame in [40] and ingoing Eddington-Finkelstein one in [10].

dual geometry at each order of late-time expansion understood as regularity of late-time-expanded curvature invariants, not necessarily the metric itself. In particular, in order to calculate Israel-Stewart relaxation time, it was required to compute the metric up to third order in the late-time expansion and demand regularity of late-time-expanded the Kretschmann scalar $R_{ABCD}R^{ABCD}$ in that order. This turned out to be impossible – even after getting rid of all power-like singularities, the Kretschmann scalar at third order contained a left-over logarithmic singularity [68]. The most straightforward and obvious interpretation *back then* was that the boost-invariant flow cannot be realized in a sector of universal dynamics of one-point function of energy-momentum tensor and some additional bulk matter fields need to be provided (or equivalently some additional operators need to acquire expectation value on gauge theory side of duality), inclusion of which would yield regular geometry [68][5]. There was only a finite number of such fields, since only zero modes of Kaluza-Klein reduction on compact manifold were of interest. However, singularities were present also in late-time-expanded more involved curvature invariants in such a way that adding a finite number of matter fields would not cure them [16]. This led to a conjecture that boost-invariant dynamics cannot be realized in gauge theories with classical gravity dual [16, 17].

On the other hand, as explained in Chapter 3, boost-invariant hydrodynamics described by velocity (3.11) and temperature (3.13) is just a particular solution of hydrodynamic equations and as such can be substituted into fluid/gravity duality metric obtained up to second order in gradients in [10]. Such geometry is manifestly regular[6] up to second order in gradients and it is hard to envisage how potential singularities would arise at higher orders of the bulk gradient expansion, though there is no explicit construction beyond those first two orders[7]. This led to a confusion whether there was something fundamentally wrong with boost-invariant flow or rather with the way bulk metric was reproduced at late time in [40] and follow-up works, the last possibility being that the condition of non-singularity as understood in [40] was somehow misleading. The aim of this Chapter based on the Letter [5] is to provide explicitly regular gravity solution dual to boost-invariant hydrodynamics up to third order in gradients utilizing ingoing Eddington-Finkelstein coordinates and explain its relation to Fefferman-Graham approach of Janik and Peschanski[8]. Manifestly regular character of constructions [5] and [91, 92] made it possible to calculate the position of the event horizon in [90] and independently in [49] completing the proof of regularity of boost-invariant bulk geometry at late time (see Chapter 5 for an extensive discussion) and assuring that the gravity dual to boost-invariant flow is a

[5]A very recent example of such reasoning is a bulk black brane-like metric, which leads to anisotropic static energy-momentum tensor of the schematic diagonal form $(\epsilon, p_{\|}, p_{\perp}, p_{\perp})$. In the absence of additional matter fields in the bulk, such geometry is known to possess naked singularity [87]. On the other hand, inclusion of non-Abelian bulk gauge field leads to a smooth solution of Einstein's equations, which in dual quantum field theory picture has anisotropic energy-momentum tensor and non-zero expectation value of vector operator [88]. Such setup is dubbed holographic p-wave superconductor [89]. The bulk system of gravity coupled to a gauge field is just one out of many possible sets of fields for which boundary energy-momentum tensor is anisotropic in global equilibrium.

[6]Here meaning the absence of terms singular at finite nonzero values of radial position measured in temperature units both in metric itself as well as in curvature expanded in gradients. Full proof of regularity requires presence of the event horizon shielding standard black brane singularity, which was only proven in later publications [90, 49] and is one of the subjects of Chapter 5.

[7]This is due to technical complexity of the problem and insufficient motivation, rather than any fundamental obstacle.

[8]See [91] and [92] for an independent construction of gravity dual to boost-invariant flow in Eddington-Finkelstein coordinates. The authors also provide a proof that gravity dual to boost-invariant flow is regular up to all orders in bulk hydrodynamic gradient expansion.

trustable framework contrary to previous claims [16, 17]. This cleared the way to calculate the effect of higher derivative corrections to the universal gravity action on some of transport properties[9] of holographic conformal field theories, which was done in [93] and [64]. Chapter 6 provides more details about [64].

4.2 Bulk construction utilizing Fefferman-Graham coordinates

As anticipated before, holographic reconstruction of spacetime can be done for an arbitrary conserved and traceless 4×4 matrix identified with boundary energy-momentum tensor. However, in vast majority of cases spacetime will have a naked singularity somewhere in the bulk signalling an unphysical holographic field theory configuration. In the case of boost-invariant dynamics the energy-momentum tensor is fully specified in terms of energy density. Functional form of the energy-momentum tensor with three a priori distinct entries on the diagonal (3.5) suggest to introduce the following metric Ansatz[10] in Fefferman-Graham variables

$$\mathrm{d}s^2 = \frac{1}{z^2}\left\{\mathrm{d}z^2 - e^{a(\tau,z)}\mathrm{d}\tau^2 + \tau^2 e^{b(\tau,z)}\mathrm{d}y^2 + e^{c(\tau,z)}\mathrm{d}x_\perp^2\right\}. \tag{4.1}$$

This metric Ansatz contains 3 functions which are subject to asymptotic AdS boundary conditions at $z=0$, so that for small z they all start as z^4 and have a regular expansion in even powers of z. More concretely, the near-boundary expansion of warp-factors a, b and c reads (note suppressed τ dependence in ϵ)

$$\begin{aligned}
a &= -\bar\epsilon\, z^4 + \left\{-\frac{1}{4\tau}\bar\epsilon' - \frac{1}{12}\bar\epsilon''\right\} z^6 + \left\{ -\frac{1}{6}\bar\epsilon^2 + \frac{1}{128\,\tau^3}\bar\epsilon' - \frac{1}{6}\tau\,\bar\epsilon\,\bar\epsilon' - \frac{1}{16}\tau^2\bar\epsilon'^2 - \frac{1}{128\,\tau^2}\bar\epsilon'' + \right.\\
&\quad \left. -\frac{1}{64\,\tau}\bar\epsilon^{(3)} - \frac{1}{384}\bar\epsilon^{(4)}\right\} z^8 + \ldots\\
b &= \left\{-\bar\epsilon - \tau\,\bar\epsilon'\right\} z^4 + \left\{-\frac{1}{3}\bar\epsilon'' - \frac{1}{12}\bar\epsilon^{(3)}\right\} z^6 + \left\{ -\frac{1}{6}\bar\epsilon^2 - \frac{1}{64\,\tau^3}\bar\epsilon' - \frac{1}{6}\tau\,\bar\epsilon\,\bar\epsilon' - \frac{1}{16}\tau^2\bar\epsilon'^2 + \right.\\
&\quad \left. +\frac{1}{64\,\tau^2}\bar\epsilon'' - \frac{1}{128\,\tau}\bar\epsilon^{(3)} - \frac{7}{384}\bar\epsilon^{(4)} - \frac{1}{384}\tau\,\bar\epsilon^{(5)}\right\} z^8 + \ldots\\
c &= \left\{\bar\epsilon + \frac{1}{2}\tau\,\bar\epsilon'\right\} z^4 + \left\{\frac{1}{8\,\tau}\bar\epsilon' + \frac{5}{24}\bar\epsilon'' + \frac{1}{24}\tau\,\bar\epsilon^{(3)}\right\} z^6 + \left\{ -\frac{1}{6}\bar\epsilon^2 + \frac{1}{256\,\tau^3}\bar\epsilon' - \frac{1}{6}\tau\,\bar\epsilon\,\bar\epsilon' + \right.\\
&\quad \left. -\frac{1}{16}\tau^2\bar\epsilon'^2 - \frac{1}{256\,\tau^2}\bar\epsilon'' + \frac{3}{256\,\tau}\bar\epsilon^{(3)} + \frac{1}{96}\bar\epsilon^{(4)} + \frac{1}{768}\tau\,\bar\epsilon^{(5)}\right\} z^8 + \ldots
\end{aligned} \tag{4.2}$$

where $\bar\epsilon$ is a rescaled energy density given by

$$\bar\epsilon = \frac{2\pi^2}{N_c^2}\epsilon\,. \tag{4.3}$$

Up to this point everything is exact and the formula (4.2) is valid for any function $\epsilon(\tau)$[11]. However in such a general setup it is a priori not known how to single out the physical $\epsilon(\tau)$

[9] More precisely shear viscosity η, relaxation time τ_Π and second order transport coefficient λ_1. Note that although the former two can be obtained from linearized hydrodynamics, the latter comes from term nonlinear in gradients and can be obtained utilizing either full fluid/gravity duality or the gravity dual to boost-invariant flow.

[10] In the rest of the text (apart from Chapter 6) AdS radius $\mathcal{L}$ is taken for simplicity to unity and can be restored using dimensional analysis.

[11] Or equivalently $\bar\epsilon\,(\tau)$.

and in [40] Janik and Peschanski proposed to focus on late-time behavior *assuming* asymptotic form of energy density to be

$$\epsilon(\tau) \sim \frac{1}{\tau^s} \tag{4.4}$$

with s constrained to lie within the range $0 < s < 4$ by positivity of energy density in any frame[12]. Keeping at each order of the near-boundary expansion (4.2) leading order asymptotics was shown in [40] to be equivalent to an introduction of a scaling variable $v = z/\tau^{s/4}$, which is kept fixed while taking large-τ limit. This observation made by Janik and Peschanski reduces Einstein's equations to a solvable, but nonlinear, system of ordinary differential equations in the scaling variable v. Although the solution of those equations exists for any s appearing in (4.4), the Kretschmann scalar $R_{ABCD}R^{ABCD}$ in the scaling limit ($\tau \to \infty$ keeping v fixed) is regular only for $s = 4/3$. Comparison with the hydrodynamic analysis of boost-invariant evolution in Chapter 3 makes it is clear, that $s = 4/3$ has an interpretation of boost-invariant perfect fluid hydrodynamics and the reasoning of Janik and Peschanski can be thus regarded as a *derivation* of nonlinear hydrodynamics of holographic conformal field theory in the boost-invariant case.

The final result of the analysis presented in [40] was the metric of the form

$$\mathrm{d}s^2_{\tau\to\infty} = \frac{1}{z^2}\left\{\mathrm{d}z^2 - \frac{\left(1-\frac{\pi^4\Lambda^4}{4}\frac{z^4}{\tau^{4/3}}\right)^2}{1+\frac{\pi^4\Lambda^4}{4}\frac{z^4}{\tau^{4/3}}}\mathrm{d}\tau^2 + \tau^2\left(1+\frac{\pi^4\Lambda^4}{4}\frac{z^4}{\tau^{4/3}}\right)\mathrm{d}y^2 + \left(1+\frac{\pi^4\Lambda^4}{4}\frac{z^4}{\tau^{4/3}}\right)\mathrm{d}\mathbf{x}^2_\perp\right\} \tag{4.5}$$

which bears a striking similarity to a boosted and dilated AdS-Schwarzschild black brane in Fefferman-Graham coordinates[13] with both boost parameter and temperature being exactly the ones of boost-invariant hydrodynamics given by (3.11) and (3.13)[14]. On the other hand, the Fefferman-Graham coordinate frame breaks down at $z = 2^{1/2}\pi^{-1}\Lambda^{-1}\tau^{1/3}$ and at this level it is impossible to give a geometrical interpretation of spacetime described by the metric (4.5) in terms of a Penrose diagram. This is precisely the reason why the surface $z = 2^{1/2}\pi^{-1}\Lambda^{-1}\tau^{1/3}$ at $\tau \to \infty$ is called "horizon-to-be" rather then genuine event horizon despite the striking resemblance of leading order boost-invariant and standard black brane metrics[15].

It should be clear that the metric (4.5) is not an exact solution of Einstein's equations, in the same way as perfect fluid hydrodynamics is not a full solution of hydrodynamic equations, and it will be corrected by gradients terms[16]. The guiding principle in both understanding the structure of subleading terms at late time, as well as in fixing integration constants appearing on the way[17] was non-singularity of the Kretschmann scalar $R_{ABCD}R^{ABCD}$[18]. In particular,

[12] Such condition is not expected to hold for a general energy density. See [40] for more extensive discussion.

[13] Given by the metric $\mathrm{d}s^2 = -\frac{\left(1-z^4\lambda^4\right)^2}{z^2\left(1+z^4\lambda^4\right)}u_\mu\, u_\nu\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu + \frac{1}{z^2}\left(1+z^4\lambda^4\right)\left(\eta_{\mu\nu}+u_\mu u_\nu\right)\mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{1}{z^2}\mathrm{d}z^2$.

[14] In a sense of asymptotic expression.

[15] Note also that locating event horizon in time-dependent setting is a very subtle issue due to its teleological nature. See Chapter 5 for more detail.

[16] Being just leading order corrections – note the presence of exponentially suppressed non-hydrodynamic tale.

[17] Most of integration constants were fixed by requiring AdS asymptotics and only one at each order (precisely the one related to expectation value of energy-momentum tensor) by non-singularity condition.

[18] As well as more complicated scalars made out of curvature; metric (4.1) itself was not demanded to be regular, which was attributed at the time of original publications to the fact, that the form of the metric depends on coordinate frame.

article [86] postulated following structure of subleading terms

$$
\begin{aligned}
a(\tau,z) &= a_0\left(\frac{z}{\tau^{1/3}}\right)+\frac{1}{\tau^{2/3}}a_1\left(\frac{z}{\tau^{1/3}}\right)+\frac{1}{\tau^{4/3}}a_2\left(\frac{z}{\tau^{1/3}}\right)+\ldots\\
b(\tau,z) &= b_0\left(\frac{z}{\tau^{1/3}}\right)+\frac{1}{\tau^{2/3}}b_1\left(\frac{z}{\tau^{1/3}}\right)+\frac{1}{\tau^{4/3}}b_2\left(\frac{z}{\tau^{1/3}}\right)+\ldots\\
c(\tau,z) &= c_0\left(\frac{z}{\tau^{1/3}}\right)+\frac{1}{\tau^{2/3}}c_1\left(\frac{z}{\tau^{1/3}}\right)+\frac{1}{\tau^{4/3}}c_2\left(\frac{z}{\tau^{1/3}}\right)+\ldots
\end{aligned}
\tag{4.6}
$$

which was later derived using again non-singularity argument in [68]. Note that the scaling variable $v=z/\tau^{1/3}$ implies that radial direction in AdS is measured in units of (asymptotic form of) boost-invariant temperature (3.10), whereas $1/\tau^{2/3}$ damping of subsequent terms has an interpretation of gravity counterpart of the boundary gradient expansion. The limiting procedure $\tau\to\infty$ in the scaling limit can be thus intuitively understood as focusing in the bulk of AdS on counterpart of IR dynamics of hCFT.

Integration constants appearing as leading order coefficients at z^4 in z^2 expansion of functions $a_i\left(z/\tau^{1/3}\right)$ are interpreted through (1.10) with coefficients in the late-time expansion of boost-invariant energy density or equivalently temperature (3.13). Einstein's equations are thus solved order by order in the scaling limit with the following structure appearing in the first two orders: in order to fix integration constants corresponding to transport properties of holographic conformal field theory at order i, Einstein's equations have to be solved up to order $i+1$ and then regularity of the Kretschmann scalar expanded up to this order in the scaling expansion

$$
R_{ABCD}R^{ABCD}=Rsq_0\left(\frac{z}{\tau^{1/3}}\right)+\frac{1}{\tau^{2/3}}Rsq_1\left(\frac{z}{\tau^{1/3}}\right)+\ldots+\frac{1}{\tau^{2(i+1)/3}}Rsq_{i+1}\left(\frac{z}{\tau^{1/3}}\right)+\ldots \tag{4.7}
$$

(more precisely regularity of $Rsq_{i+1}\left(z/\tau^{1/3}\right)$ term at $z/\tau^{1/3}=2^{1/2}\pi^{-1}\Lambda^{-1}$) fixes the value of the coefficient in question. This procedure was used in [94] to derive viscous correction to perfect fluid metric, which takes the form

$$
\begin{aligned}
a_1(v) &= \frac{2\eta_0}{\Lambda}\frac{\left(\frac{12}{\pi^4\Lambda^4}+v^4\right)v^4}{\frac{16}{\pi^8\Lambda^8}-v^8},\\
b_1(v) &= -\frac{2\eta_0}{\Lambda}\frac{v^4}{\frac{4}{\pi^4\Lambda^4}+v^4}+\frac{2\eta_0}{\Lambda}\log\frac{\frac{4}{\pi^4\Lambda^4}-v^4}{\frac{4}{\pi^4\Lambda^4}+v^4},\\
c_1(v) &= -\frac{2\eta_0}{\Lambda}\frac{v^4}{\frac{4}{\pi^4\Lambda^4}+v^4}-\frac{\eta_0}{\Lambda}\log\frac{\frac{4}{\pi^4\Lambda^4}-v^4}{\frac{4}{\pi^4\Lambda^4}+v^4},
\end{aligned}
\tag{4.8}
$$

where v is a scaling variable $v=z\,\tau^{-1/3}$ and η_0 is related to shear viscosity of holographic gauge theory by (3.14). Evaluating the late-time-expanded the Kretschmann scalar (4.7) at second order, done in [86], leads to the following structure

$$
R_{ABCD}R^{ABCD}=\text{regular terms}+\frac{1}{\tau^{4/3}}\frac{\text{Polynomial in }\eta_0\text{ and }v}{\left(\frac{4}{\pi^4\Lambda^4}-v^4\right)^4\left(\frac{4}{\pi^4\Lambda^4}+v^4\right)^6}+\mathrm{O}\left(\frac{1}{\tau^2}\right), \tag{4.9}
$$

where the power-like singularity at $v = 2^{1/2}\pi^{-1}\Lambda^{-1}$ is cancelled only for $\eta_0 = 1/3\pi$. Using holographic results (1.21) and (1.24), $\eta_0 = 1/3\pi$ can be translated into $\eta/s = 1/4\pi$, which is the expected number, signalling self-consistency of the approach [86][19].

Note, that although first order formulas are singular at $v = 2^{1/2}\pi^{-1}\Lambda^{-1}$, by taking τ sufficiently large one can nevertheless make sense of this expansion until arbitrary close to $v = 2^{1/2}\pi^{-1}\Lambda^{-1}$. However, the point $v = 2^{1/2}\pi^{-1}\Lambda^{-1}$ *is not covered* by the Fefferman-Graham coordinates and so checking non-singularity of the metric there is not really guaranteed to work. Despite that, the condition of non-singularity of the Kretschmann scalar at zeroth, first and second order of bulk late-time expansion worked well and shear viscosity derived in such way agreed, as anticipated before, with results obtained by other means. On the other hand, in the bulk calculation of the Israel-Stewart relaxation time a logarithmic singularity of the form $\log\left(2^{1/2}\pi^{-1}\Lambda^{-1} - v\right)$ showed up at the third order (so at τ^{-2}) in the expansion of the Kretschmann scalar [68], which questioned the validity of the whole approach. The perfect agreement between the gauge theory results delivered by Janik and Peschanski framework and by other methods, which yield explicitly regular gravity dual (in particular fluid/gravity duality [10]), suggests that there might be a subtlety hidden in the non-singularity condition adopted in [40] and in the follow-up works, rather than a naked singularity spoiling this background. In the rest of the Chapter it is shown how to obtain a manifestly regular gravity dual to boost-invariant flow up to third order in gradients in the ingoing Eddington-Finkelstein coordinate frame, which settles down the issue whether it is possible to realize the boost-invariant expansion in planar strongly coupled hCFTs. Then a singular perturbative coordinate transformation from the regular solution in the Eddington-Finkelstein frame to the Fefferman-Graham chart is presented (and vice versa) yielding *precisely* the metric found in [68]. The physical interpretation of this result in given in the last Section.

4.3 Bulk construction utilizing Eddington-Finkelstein coordinates

Following the ideas of fluid/gravity duality introduced in [10] and reviewed in Chapter 2, boost-invariant perfect fluid flow can be obtained locally from the 5-dimensional boosted black brane solution expressed in ingoing Eddington-Finkelstein coordinates

$$ds^2 = -2u_\mu \mathrm{d}x^\mu \mathrm{d}r - r^2\left(1 - \frac{\pi^4 T^4}{r^4}\right)u_\mu u_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu + r^2\left(\eta_{\mu\nu} + u_\mu u_\nu\right)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{4.10}$$

where u^μ is the boost velocity parameter and T the temperature. The key ingredient of this approach is the introduction of the ingoing Eddington-Finkelstein coordinates. At the boundary this chart reduces to usual Minkowski coordinates. Fluid/gravity duality maps long-wavelength (IR) universal dynamics of the one-point function of energy-momentum, the object naturally defined in the vicinity of the boundary of AdS spacetime, into the bulk by decreasing r while keeping the Eddington-Finkelstein time and spatial coordinates fixed [69]. For a Bjorken expansion it is natural to use proper time τ instead of the usual Minkowski time, so an analogous Eddington-Finkelstein type proper time coordinate $\tilde{\tau}$ is introduced. Specifically, $u = \partial_\tau$ at the boundary, but is now taken as $u = \partial_{\tilde{\tau}}$ in the bulk of AdS spacetime.

[19] Note different convention than in original works [94, 86] – keeping constant Λ unspecified in all expressions instead of using $\Lambda = \frac{\sqrt{2}}{3^{1/4}\pi}$ as in [40].

Furthermore, for a boost invariant flow the temperature T is asymptotically proportional to $\tau^{-1/3}$, which gives

$$\mathrm{d}s^2 = -r^2(1-\frac{\pi^4\Lambda^4}{\tilde{\tau}^{4/3}r^4})\mathrm{d}\tilde{\tau}^2 + 2\mathrm{d}\tilde{\tau}\mathrm{d}r + r^2\tilde{\tau}^2\mathrm{d}y^2 + r^2\mathrm{d}x_\perp^2. \tag{4.11}$$

Note that in boost-invariant ingoing Eddington-Finkelstein coordinates $\tilde{\tau} = \mathrm{const}$ is an equation for ingoing null geodesics. This implies that surfaces of constant proper time must eventually cross the future event horizon if present in the bulk. In particular, absence of singularities at finite value of radial variable measured in units of energy density or temperature at given time instance $\tilde{\tau}$ should be sufficient to have a regular geometry at any later time. Such a nice feature is not present in the Fefferman-Graham chart, but it is nevertheless demonstrated that metric (4.11) is related to the Janik-Peschanski metric (4.5) by a *singular* coordinate transformation

$$\begin{aligned}\tilde{\tau} &= \tau\left\{1-\frac{1}{\tau^{2/3}}\left[\frac{1}{4\,\Lambda}+\frac{1}{2\,\pi\,\Lambda}\arctan\left(\frac{r\cdot\tau^{1/3}}{\pi\,\Lambda}\right)+\frac{1}{4\,\pi\,\Lambda}\log\frac{r\cdot\tau^{1/3}-\pi\,\Lambda}{r\cdot\tau^{1/3}+\pi\,\Lambda}\right]\right\},\\ r &= \frac{1}{z}\cdot\sqrt{1+\frac{\pi^4\Lambda^4}{4}\frac{z^4}{\tau^{4/3}}}.\end{aligned} \tag{4.12}$$

which works provided keeping scaling limit in both coordinate frames and neglecting terms subleading when τ and $\tilde{\tau}$ are both large. Note also that the relation between τ and $\tilde{\tau}$ is singular when $z = 2^{1/2}\pi^{-1}\Lambda^{-1}\tau^{1/3}$. This is precisely the locus where the singularities found in [68] were encountered. In particular, the perturbative coordinate transformation (4.12) works up to arbitrarily close to the point $v = 2^{1/2}\pi^{-1}\Lambda^{-1}$, but exactly at that point all order resummation is required from the formal point of view. Compare this with a singular coordinate transformation from the Fefferman-Graham to the ingoing Eddington-Finkelstein coordinates in the case of eternal AdS-Schwarzschild black brane. The singularity in the latter transformation is required to extend the patch covered by the Fefferman-Graham coordinates and go past the horizon. Note that since this geometry is static, the position of the singularity in the transformation does not change in time. The question whether such a perturbative transformation from the Fefferman-Graham to Eddington-Finkelstein coordinate frame (or the other way around) works up to the third order was the main problem addressed in the original publication. To achieve this, an explicitly regular gravity background in the ingoing Eddington-Finkelstein coordinates dual to third order boost-invariant hydrodynamics is computed and then a singular perturbative coordinate transformation is carefully built order by order mapping therefore both solutions.

The metric (4.11) is not an exact solution of Einstein's equations with negative cosmological constant – there are subleading corrections coming from derivatives of the velocity u and temperature T. They correspond to the gradient expansion of the boundary energy-momentum tensor [44]. The apparent curvature singularities in AdS encountered in [68] appear at third order in the large τ (gradient) expansion. It is difficult to check what happens for a general flow at this order of fluid/gravity duality. However the situation is much simpler for a boost-invariant flow, since all the symmetries can be imposed from the outset, as has been done in Fefferman-Graham frame. This leads to the following ansatz for the metric[20]

$$\mathrm{d}s^2 = G_{MN}\mathrm{d}x^M\mathrm{d}x^N = -r^2\tilde{A}(\tilde{\tau},r)\mathrm{d}\tilde{\tau}^2 + 2\mathrm{d}\tilde{\tau}\mathrm{d}r + (1+r\,\tilde{\tau})^2\,e^{\tilde{b}(\tilde{\tau},r)}\mathrm{d}y^2 + r^2e^{\tilde{c}(\tilde{\tau},r)}\mathrm{d}x_\perp^2. \tag{4.13}$$

[20]Factor $(1+r\,\tilde{\tau})^2$ at $\mathrm{d}y^2$ is to ensure that the limit $T\to 0$ leads to an empty AdS spacetime, as explained in [91].

Introducing new scaling variable $\tilde{v} = r \cdot \tilde{\tau}^{1/3}$ in analogy with what is done in [40, 94, 86, 68, 16, 17] one obtains the natural expansion of the metric components in $\tilde{\tau}^{-2/3}$ on the gravity side

$$
\begin{aligned}
\tilde{A}(\tilde{\tau}, r) &= \sum_{k\geq 0} \tilde{A}_k(\tilde{v})\, \tilde{\tau}^{-2k/3}, \\
e^{\tilde{b}(\tilde{\tau}, r)} &= \tilde{B}(\tilde{v}) \exp\left(\sum_{k>0} \tilde{b}_k(\tilde{v}) \tilde{\tau}^{-2k/3}\right), \\
e^{\tilde{c}(\tilde{\tau}, r)} &= \tilde{C}(\tilde{v}) \exp\left(\sum_{k>0} \tilde{c}_k(\tilde{v}) \tilde{\tau}^{-2k/3}\right).
\end{aligned}
\tag{4.14}
$$

To obtain a uniform expansion of the Einstein's equations $E_{MN} \equiv \mathcal{R}_{MN} - \frac{1}{2}\mathcal{R}\, G_{MN} - 6\, G_{MN} = 0$ one needs to rescale them (see [86]) according to $\hat{E} = \left(\tilde{\tau}^{2/3} E_{\tilde{\tau}\tilde{\tau}}, E_{\tilde{\tau}r}, \frac{1}{\tilde{\tau}^{2/3}} E_{rr}, \frac{1}{\tilde{\tau}^{4/3}} E_{yy}, \tilde{\tau}^{2/3} E_{x_\perp x_\perp}\right)$. This leads to

$$
\hat{E}(\tilde{\tau}, r) = \hat{E}_0(r \cdot \tilde{\tau}^{1/3}) + \frac{1}{\tilde{\tau}^{2/3}} \hat{E}_1(r \cdot \tilde{\tau}^{1/3}) + \frac{1}{\tilde{\tau}^{4/3}} \hat{E}_2(r \cdot \tilde{\tau}^{1/3}) + \frac{1}{\tilde{\tau}^{2}} \hat{E}_3(r \cdot \tilde{\tau}^{1/3}) + \ldots \tag{4.15}
$$

The curvature invariants (e.g. $\mathcal{R}_{MNOP}\mathcal{R}^{MNOP}$) defined recursively in [16] can be likewise expanded. The crucial difference between the present approach and the one introduced in [40] is that the expansion parameter involves $\tilde{\tau}$ instead of τ. Einstein's equations can be solved order by order in $\tilde{\tau}^{-2/3}$ expansion starting from

$$
\begin{aligned}
\tilde{A}_0\,(\tilde{v}) &= 1 - \frac{\pi^4 \Lambda^4}{\tilde{v}^4}, \\
\tilde{B}\,(\tilde{v}) &= \tilde{C}\,(\tilde{v}) = 1,
\end{aligned}
\tag{4.16}
$$

which simply reproduces the boosted black brane solution (4.11). Thus the zeroth order solution entails large but finite $\tilde{\tau}$. The singularity at $r = 0$ (or equivalently at $\tilde{v} = 0$) should be shielded by an event horizon, which is indeed the case as shown in Chapter 5 and references [90, 49]. Note also that the metric in Eddington-Finkelstein coordinates is well-defined even when $\tilde{A}$ is zero due to the presence of off-diagonal term, which is a crucial difference between those coordinates and Fefferman-Graham frame.

Gravity dual of the boost-invariant gradient expansion

The equations of motion (4.15) at a given order k are a system of ordinary second order differential equations for the 3 functions $\tilde{A}_k\,(\tilde{v})$, $\tilde{b}_k\,(\tilde{v})$ and $\tilde{c}_k\,(\tilde{v})$. Each solution involves two integration constants. On the other hand, two of the equations of motion are constraints. At each order $k > 0$ one of the constraints fixes one of the integration constants appearing at that order, and the other one fixes an integration constant left undetermined at order $k-1$. Out of 4 remaining integration constants, 3 can be fixed order by order by imposing asymptotic AdS behavior at infinity and metric regularity in the bulk (up to the usual black brane singularity at $\tilde{v} = 0$ [10]) and 1 is related to residual diffeomorphism $r \to r + f\,(\tilde{\tau})$ preserving the form of the metric Ansatz (4.13) as spotted in [91]. It turns out that the potential singularity is located only at $\tilde{v} = \pi\,\Lambda$, thus the functions $A_k\,(\tilde{v})$, $b_k\,(\tilde{v})$ and $c_k\,(\tilde{v})$ must remain finite as $\tilde{v} \to \pi\,\Lambda$. The residual diffeomorphism invariance [91] preserved by the Ansatz (4.13) can effectively be fixed by requiring that $\tilde{A}_k\,(\tilde{v}) = \mathrm{O}\,(v^4)$ for $k > 0$. Furthermore asymptotic AdS behavior of the metric requires that functions $\tilde{b}_k$ and $\tilde{c}_k$ vanish as $\tilde{v} \to \infty$ (in the late proper time regime).

These conditions together with the constraints fix 5 of the 6 integration constants at a given order $k > 0$ and lead to a regular metric with no poles or logarithmic singularities apart from $v = 0$. As an example, the first order solution (dual to viscous hydrodynamics [94, 86]) reads

$$\begin{aligned}\tilde{A}_1(\tilde{v}) &= \frac{2\,\pi^3\Lambda^3}{3\,\tilde{v}^4} + \frac{2\,\pi^4\Lambda^4}{3\,\tilde{v}^5},\\ \tilde{b}_1(\tilde{v}) &= \frac{1}{3\,\pi\,\Lambda}\Big\{\pi - \frac{4\pi\Lambda}{\tilde{v}} - 2\arctan\left(\frac{\tilde{v}}{\pi\Lambda}\right) + \log\left(1 + \frac{\tilde{v}^2}{\pi^2\Lambda^2}\right)\\ &\quad +2\log\left(1 + \frac{\tilde{v}}{\pi\,\Lambda}\right) - 4\log\left(\frac{\tilde{v}}{\pi\Lambda}\right)\Big\}\end{aligned} \tag{4.17}$$

with $\tilde{c}_1(\tilde{v}) = -\tilde{b}_1(\tilde{v})\,/\,2$. Higher order formulae (up to the third order) are quite lengthy, but explicitly regular, and can be found in a `Mathematica` notebook available online[21]. Moreover, holographic renormalization correctly reproduces the energy density for the boost-invariant flow up to second order in derivatives [68, 12]. Using fourth order equations of motion in Eddington-Finkelstein coordinates, third order contribution to boost-invariant energy density could be obtained and the result is presented in [49]. However, precise interpretation of third order energy density coefficient in terms of third order transport properties is (as yet) unknown, since there was no clear motivation to pursue the construction of relativistic hydrodynamics up to that order.

Absence of singularities and relation to Fefferman-Graham coordinates

The assumption of non-singularity of coefficients of curvature invariants in the Fefferman-Graham late proper time expansion was a crucial ingredient of [86, 68] needed to establish some of transport properties of $\mathcal{N} = 4$ SYM plasma and, as now understood, of universal sector of dynamics of any 3+1-dimensional hCFT [35]. Fluid/gravity duality-based holographic approach to boost-invariant flow introduced in the Letter [18] and presented in this Chapter starts from a manifestly regular metric in the leading order (no logarithmic and power-like singularities at $\tilde{v} = \pi\,\Lambda$) and produces regular solutions up to the third order. Further development by other authors [91] led to the conclusion that gravity dual to boost-invariant flow in ingoing Eddington-Finkelstein coordinates is perturbatively finite to all orders in the late-time expansion (apart from standard black brane singularity at $\tilde{v} = 0$, which is however shielded by an event horizon[22]). Since in the present approach the components of the metric, its inverse, as well as their derivatives are regular, all curvature invariants are non-singular. Indeed, from (4.13) it follows that the non-vanishing components of the inverse are $G^{rr} = r^2 A(\tilde{\tau}, r)$, $G^{r\tilde{\tau}} = 1$, $G^{yy} = r^{-2}\tilde{\tau}^{-2}e^{-\tilde{b}(\tilde{\tau},r)}$, $G^{\perp\perp} = r^{-2}e^{-\tilde{c}(\tilde{\tau},r)}$. If $A(\tilde{\tau}, r)^{-1}$ had been present, singularities would have appeared as a consequence of (4.16).

It is very natural to ask how these results are related to those obtained using the original approach of [40, 68] utilizing Fefferman-Graham coordinate chart. The simplest, and in fact realized, answer would be that they are related by a (*singular*) coordinate transformation order by order in the large proper-time expansion. Such transformation generalizing (4.12) takes the

[21] `http://th.if.uj.edu.pl/~heller/boost_ver2.nb`

[22] This was however proven only up to third order of boundary gradient expansion [49].

form

$$\tilde{\tau} = \tau\Big(t_0(\frac{z}{\tau^{1/3}}) + \frac{1}{\tau^{2/3}}t_1(\frac{z}{\tau^{1/3}}) + \frac{1}{\tau^{4/3}}t_2(\frac{z}{\tau^{1/3}}) + \frac{1}{\tau^2}t_3(\frac{z}{\tau^{1/3}}) + \ldots\Big) \tag{4.18}$$

$$r = \frac{1}{z}\Big(r_0(\frac{z}{\tau^{1/3}}) + \frac{1}{\tau^{2/3}}r_1(\frac{z}{\tau^{1/3}}) + \frac{1}{\tau^{4/3}}r_2(\frac{z}{\tau^{1/3}}) + \frac{1}{\tau^2}r_3(\frac{z}{\tau^{1/3}}) + \ldots\Big) \ . \tag{4.19}$$

and it is straightforward (though tedious) to determine coefficients explicitly; the results up to third order are available online in the aforementioned `Mathematica` notebook. The transformation (4.18) defines a map between the two approaches and explains how it was possible that energy density obtained in [68] and in previous works on the subjects was correct despite apparent singularities in the bulk. An analogous coordinate transformation was considered later in a gravity dual to non-conformal hydrodynamics [95]. The main motivation of those authors was that the holographic renormalization procedure was derived for their background in Fefferman-Graham-like coordinates, whereas to check regularity of the geometry they needed coordinate transformation to Eddington-Finkelstein-like frame anyway. Note also an explicit construction of gravity dual to a general solution of hydrodynamics up to first order in Fefferman-Graham coordinates [96].

The coordinate transformation (4.18) might be understood as a resummation of metric coefficients containing singularities into a regular solution, but now expressed in the Eddington-Finkelstein chart (e.g. compare (4.8) with (4.17)). It would be very interesting to study numerically the holographic dual to boost-invariant flow starting from some regular initial data using some well-behaving coordinate chart or utilizing the Chesler-Yaffe framework [21, 67] and follow on Penrose diagram lines of constant Fefferman-Graham chart. It would be also interesting to see the full form of numerically obtained geometry transformed to Fefferman-Graham chart to understand where precisely Fefferman-Graham result breaks down due to apparent singularities.

4.4 Current understanding of Fefferman-Graham scaling variable trick

The current understanding of Fefferman-Graham procedure presented above in the light of Letter [18] is that Fefferman-Graham *gradient-expanded* metric (4.6) is a valid description sufficiently close to the boundary at very late time, but it ceases to be correct in the vicinity of the “horizon-to-be”[23]. In particular, demanding non-singularity of curvature invariants at $v = 2^{1/2}\pi^{-1}\Lambda^{-1}$ is not really correct either[24], but produces correct results close to the boundary leading to a physical energy density. Singular coordinate transformation presented in subsequent section explains this, signalling that demanding absence of power-like singularities in the scaling expansion of Riemann squared in Fefferman-Graham chart is equivalent to non-singularity condition at the *genuine event horizon*[25] in Eddington-Finkelstein coordinates, where the bulk gradient expansion is valid all the way up to the horizon itself. It is beyond any doubts that Fefferman-Graham approach does not lead to a nice geometric picture in which the causal structure of spacetime is transparent. On the other hand, the singularity in

[23] Late-time-expanded metric coefficients are singular leading to a break down of gradient expansion.

[24] Because this is the point where coordinate frame breaks down – gradient corrections to the metric blow up there.

[25] see Chapter 5 for a discussion on event horizon in the gravity dual to boost-invariant flow

the Fefferman-Graham late-time-expanded square of Riemann tensor (as well as higher order curvature invariants) is a singularity of the expansion scheme, rather than a genuine curvature singularity (being a naked singularity) contrary to the claims in [35]. As stressed before, the singular coordinate transformation presented in [18] and elucidated in this Chapter can be understood as a resummation of the Fefferman-Graham expansion leading to a regular result.

For a suggestive analogy one can consider linearized perturbation in the eternal AdS black hole background given in Fefferman-Graham coordinate frame. After Fourier decomposition perturbation $\delta\phi$ is given by some function depending only on the radius, modulated by a standard oscillating phase factor

$$\delta\phi = \phi(z) e^{-i\,\omega\,t + i\,\vec{k}\cdot\vec{x}}\,. \tag{4.20}$$

The causality of boundary quantum field theory requires that this perturbation is ingoing into the horizon (falls into the black brane – see [73] and references therein). In particular, close to the horizon in Fefferman-Graham-like coordinates (so near $z = z_0$) this perturbation should behave as

$$\phi\left(z\right) = \left(z_0 - z\right)^{i\frac{\omega}{3}\,z_0} \left\{\text{regular terms}\right\} \tag{4.21}$$

or after putting it into an exponent $\log\left(z - z_0\right)$. This logarithmically singular term can be understood as a part of coordinate transformation to ingoing Eddington-Finkelstein, so that close to the horizon the perturbation in those coordinates takes the form

$$\delta\phi = e^{-i\,\omega\,t_{EF} + i\,\vec{k}\cdot\vec{x}} + \mathrm{O}\left(\left(r - r_0\right)^2\right). \tag{4.22}$$

Logarithmic singularity present in the late-time-expanded in Fefferman-Graham coordinates should have a similar interpretation – it is a contribution, which shifts the coordinates, so that one ends up on event horizon, where everything is regular if expressed in Eddington-Finkelstein, Kruskal or other nice coordinate frame.

Chapter 5

Entropy production in gravity dual to boost-invariant flow

5.1 Area theorem as second law of thermodynamics of hCFTs

A generalization of the notion of entropy to hydrodynamics is provided by a hydrodynamic entropy current J^{μ}, which is constructed order by order in the gradient expansion (see Section 2.3 in Chapter 2 for an introduction). Such construction involves an ambiguity [69, 13] – at second order a 4-parameter family of currents[1] was identified in [69]. Subsequently some important considerations on this topic have appeared in the literature [13], which suggest that it may be possible to reduce this freedom to just a single parameter. In the light of AdS/CFT duality it is natural to ask if one can understand the origins and implications of possible ambiguities in a hydrodynamic entropy current on the gravity side of the correspondence.

At the root of this question lies the identification of the area increase theorems in general relativity (see [48] for a brief review) with the second law of thermodynamics of dual gauge theory. Thus if one hopes to understand this ambiguity on the gravity side, the first step is to carefully examine and understand how horizon areas increase. In doing this, one may build on the experience gained in the study of standard black holes in asymptotically flat $(3+1)$-dimensional spacetime [97]. Generalizing these results to 5-dimensional AdS spacetime is straightforward, but deep questions encountered earlier remain.

The most "conservative" definition of entropy identifies it with the area of a spatial section of the event horizon. This notion has the drawback of being a teleological concept and in the context of AdS/CFT duality leads to acausality in the field theory [21, 90] (see next Section for a brief summary of [21]). In classical general relativity, the problems raised by the global character of the event horizon have led to alternative, quasilocal notions of black objects including trapping [98], isolated [99, 100, 101], and dynamical [102, 103] horizons. These programs are closely related to each other and motivated by historical ideas about trapped surfaces [104] and apparent horizons [97].

Quasilocal horizons have a problem of their own, namely non-uniqueness: unlike event horizons, in dynamical spacetimes containing a black object, there are many possible time-evolved

[1] In general there might be a 5-parameter ambiguity, but one of these parameters is connected with parity-odd effects. This Thesis considers only the parity-even case.

quasilocal horizons. In the case of apparent horizons, each foliation of the spacetime will give rise to a different time-evolved horizon [97] while trapping/dynamical horizons are also subject to deformations (see, for example, [105] [106]). Though this is generally thought to be a bad thing, in the context of AdS/CFT it is natural to suspect that the ambiguity of the hydrodynamic entropy current may be related to ambiguities of this type.

In this Chapter, based on [49], it is shown that non-uniqueness of time-evolved apparent horizons is not the source of the ambiguity in the dual hydrodynamic entropy current. Instead, the already existing uncertainty as to whether the event or (time-evolved) apparent horizon determines the entropy is exploited to advocate a more general approach mimicking in the bulk the phenomenological construction of the boundary entropy current. In the spirit of the membrane paradigm [107] more general "horizons" are considered, which are made up from families of (not necessarily trapped) codimension-two surfaces that satisfy properties such as area increase, asymptoting to the correct equilibrium limit, and being "almost" (in the appropriate sense, for details see original publication [49]) apparent horizons.

It is conjectured here that the ambiguity in the above definition of black brane entropy corresponds to the known ambiguity of the hydrodynamic entropy current in the appropriate regime on the gravity side. Verifying this claim in the general case of fluid/gravity duality [35] is a subject of ongoing work [108] and here this question is explored in the gravity dual to Bjorken flow at late proper time. Besides being tractable, this geometry has the important feature that there is a *unique* time-evolved apparent horizon consistent with symmetries of boundary flow, which implies that the ambiguity in the hydrodynamic entropy current (in case of Bjorken flow) cannot be interpreted as a consequence of slicing dependence (see Chapter 3 for details of entropy current in boost-invariant hydrodynamics). The rest of the Chapter follows along the lines of [49].

5.2 Various notions of horizons in the bulk

5.2.1 A brief review of black objects and their horizons

Black objects in general relativity can be defined in various ways. While different definitions of black holes or black branes generally agree for stationary black objects, they diverge away from equilibrium. In particular, they identify different surfaces in spacetime as the boundary of the black hole or black brane region and these surfaces have different surface areas. Since thermodynamics of holographic gauge theory is identified with mechanics of black objects on the gravity side of the correspondence, different surface areas would suggest different values of entropy. In other words, gravity has different notions of entropy *understood* as the quantity agreeing in equilibrium limit with thermodynamic entropy and obeying the second law of thermodynamics.

Event horizons: causally defined black objects

Causally defined black holes or black branes and their event horizons are non-local objects. Not very precise, but suggestive enough for the purposes of this Chapter, a paraphrased version of the standard definition of a black object is that of a region of spacetime from which nothing can ever escape [97]. The non-local character of black objects arise precisely from the concepts of "ever" and "escape". In general, in order to identify a black hole or black brane region

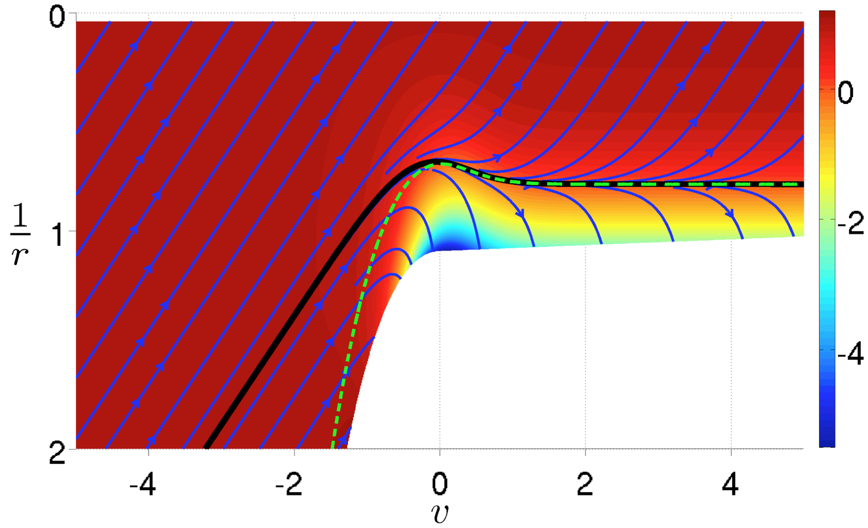


Figure 5.1: Figure taken from original reference showing outgoing radial null geodesics propagating through the geometry of [21]. v is the ingoing Eddington-Finkelstein time coinciding at the boundary with time t and r is a radial direction in AdS with $\frac{1}{r} = 0$ corresponding to the boundary of AdS. The boundary metric is significantly time-dependent between approximately $t = v = -1.7$ and $t = v = 1.7$. Blue lines show outgoing radial null geodesics, black line – the event horizon, green dashed line – the time-evolved apparent horizon (more precisely these are constant $x_{\|}$, constant $\mathbf{x}_{\perp}$ cross sections of relevant horizons). The color gradients on the plot are of no relevance for the discussion here. Note that e.g. at $v = -2$ there is no way in distinguishing light ray forming the event horizon from any other outgoing null geodesic, which makes its teleological nature transparent.

one must essentially sit at infinity and wait forever to make sure that all escaping signals are identified and further that those that initially look like they might escape really do make it to infinity, see discussion below and Figure 5.1 for a concrete example. Equivalently (but more rigorously) the black object is the complement of the causal past of future null infinity. The boundary of this region is the congruence of null geodesics known as the *event horizon*.

To better understand the teleological character of event horizon in the context of AdS/CFT correspondence, consider the following problem addressed by Chesler and Yaffe in [21]. The idea of [21] is to start with a patch of vacuum AdS spacetime and for some period of time make the boundary metric time-dependent in spatially anisotropic fashion

$$\mathrm{d}s_4^2 = \mathrm{d}t^2 + e^{B_0(t)}\mathrm{d}\mathbf{x}_{\perp}^2 + e^{-2B_0(t)}\mathrm{d}x_{\|}^2 \tag{5.1}$$

with B_0 changing rapidly over some period of time (like e.g. Heaviside step function). On the gauge theory side, the time-dependent background metric will source the energy-momentum tensor operator out of its vacuum expectation value and such excited state will eventually isotropize to form an equilibrated plasma configuration. The dual gravitational picture of this phenomenon is that the time-dependent boundary metric sources gravitational waves collapsing in the bulk of AdS and such dynamical geometry finally relaxes to become a patch of AdS-Schwarzschild black brane. The primary motivation for the authors of [21] was to estimate the isotropization time for strongly coupled non-Abelian plasmas, but an additional output of those studies was a very nice AdS/CFT example of teleological nature of the event horizon.

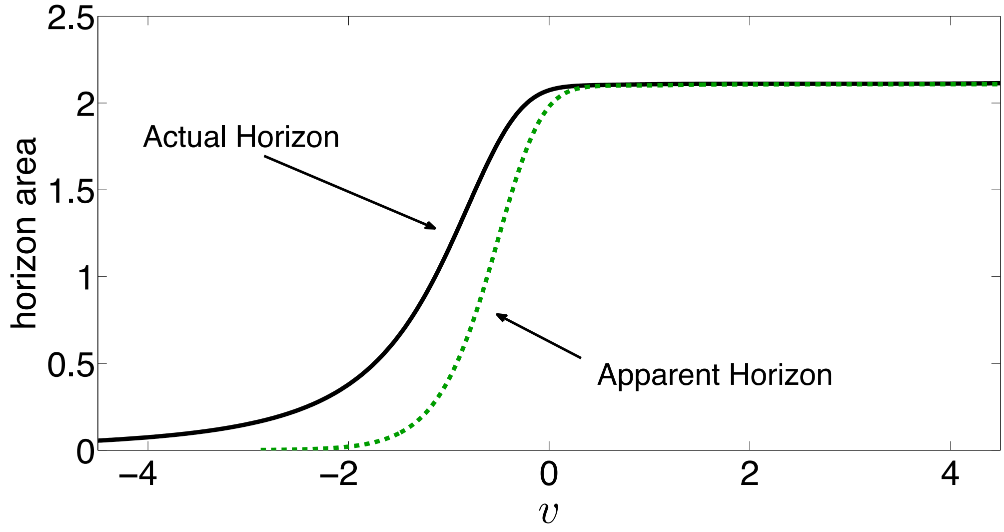


Figure 5.2: Figure taken from original reference showing areas of constant v cross sections of event (called here actual) and time-evolved apparent horizons as a function of time v in the geometry of [21]. Note that both notions are non-decreasing, although the area of the event horizon started to expand before (in the sense of ingoing Eddington-Finkelstein time) time-dependent boundary metric took the dual gauge theory out of its vacuum. The expresses again teleological aspect of causal definition of black object. Note also that the area of time-evolved apparent horizon start to change precisely when boundary metric seizes to be static and coincide with the area of the event horizon in the far future when geometry is time-independent.

The authors of [21] used an ingoing Eddington-Finkelstein time coordinate, so that an ingoing radial null signal propagates instantaneously into the bulk of AdS. Consider now Figure 5.1, which shows outgoing radial null geodesics propagating towards the boundary of asymptotically locally AdS spacetime of interest. A priori (before the geometry relaxes to be close to a patch of standard black brane spacetime) it is not possible to tell which geodesics will make it to infinity and which will hit the singularity sitting at $r = 0$. There is however a single radial null geodesic which neither reaches the boundary nor dwells into the interior of black brane – this is the one which spans the event horizon of the solution. Deciding which one it is requires the knowledge of the whole spacetime in advance. Locating the event horizon was possible in this particular case, because the geometry relaxed rather quickly to a patch of AdS-Schwarzschild solution with a static event horizon. In other words, in the context of [21] it is known a priori that no matter nor gravitational waves will traverse the horizon shortly after boundary becomes static again.

The area of spatial sections of an event horizon is always non-decreasing, which, as anticipated before, is one of the reasons why the gravitational notion of entropy has been traditionally associated with it. Consider now Figure 5.2 which shows how the area of spatial sections of the event horizon (actual horizon) in the geometry of interest evolve in time, making the teleological nature of the event horizon transparent. Note that the area of the event horizon started to increase in the past in anticipation of future events, before the boundary metric began to be time-dependent. Thus if one wants to associate the entropy density of the dual gauge theory with the area element of the event horizon, one is immediately led to acausal behavior. On the other hand, the same Figure shows time dependence of the area element of time-evolved apparent horizon, which evolves causally and starts to expand only in reaction to the arrival of gravitational waves (or matter) crossing it. This implies that in the context of AdS/CFT correspondence quasilocal notions of horizon reviewed in the following Section, should be taken rather seriously.

Apparent horizons: geometrically defined black objects

Quasilocal definitions of black objects and their boundaries leave aside the causal structure of spacetime and instead focus on the strong gravitational fields characterized by the existence of *trapped surfaces*. For regular $(3+1)$-dimensional astrophysical black holes, trapped surfaces are closed (of topology of a sphere) and spacelike two-surfaces which have the property that all families of null geodesics that intersect them orthogonally must converge into the future. To understand this intuitively, consider the following standard example. The starting point is a transparent spherical shell that is covered with light bulbs and sitting in empty space. Then if the bulbs are quickly turned on and then off again, two spherical light fronts will be generated – an outwards moving one that expands in area and an inwards moving one that contracts. By contrast, if the shell is transported so that it lies inside a Schwarzschild black hole, concentric with the horizon and enclosing the singularity, then both light fronts will fall towards the center of the black hole and contract in area. The notion of trapped surface have a simple generalization to AdS spacetimes where instead of topology of sphere it is rather a plane. Again consider Figure 5.1 where some outward and inward (those are not depicted since they move along the trajectories of v = constant) congruences of null geodesics contract in area in the RHS part of the Figure below black line denoting the event horizon. This depicts the key characteristic of a trapped surface.

More mathematically, if ℓ^A and n^A are respectively the outward and inward future pointing null normals to a codimension-two surface S then one can write

$$\theta_{(\ell)} < 0 \text{ and } \theta_{(n)} < 0\,, \tag{5.2}$$

where $\theta_{(\ell)}$ and $\theta_{(n)}$ are the *expansions* of the null normals defined as

$$\theta_{(\ell)} = \tilde{q}^{AB}\nabla_A \ell_B \text{ and } \theta_{(n)} = \tilde{q}^{AB}\nabla_A n_B \tag{5.3}$$

with a projector $\tilde{q}_{AB}$

$$\tilde{q}_{AB} = g_{AB} + \ell_A n_B + n_A \ell_B \tag{5.4}$$

being the induced (spacelike) metric on the two-dimensional surface (here ℓ^a and n^b are cross-normalized to -1: $\ell_A n^A = -1$). Alternatively given outward and inward congruences of null geodesics which have tangents ℓ^A and n^A on S one can show that

$$\sqrt{\tilde{q}}\,\theta_{(\ell)} = \frac{1}{2}\mathcal{L}_\ell\sqrt{\tilde{q}} \quad \text{and} \quad \sqrt{\tilde{q}}\,\theta_{(n)} = \frac{1}{2}\mathcal{L}_n\sqrt{\tilde{q}}\,, \tag{5.5}$$

where $\sqrt{\tilde{q}}$ is the area element on S and $\mathcal{L}$ indicates is the Lie derivative operator. It is clear then that the sign of the expansion determines whether the congruence is expanding or contracting in area.

More generally, given the energy conditions the mere existence of a trapped surface implies both a singularity somewhere inside it [104] and assuming cosmic censorship that it is necessarily contained in a causal black hole or black brane and so event horizon [97]. Indeed, for

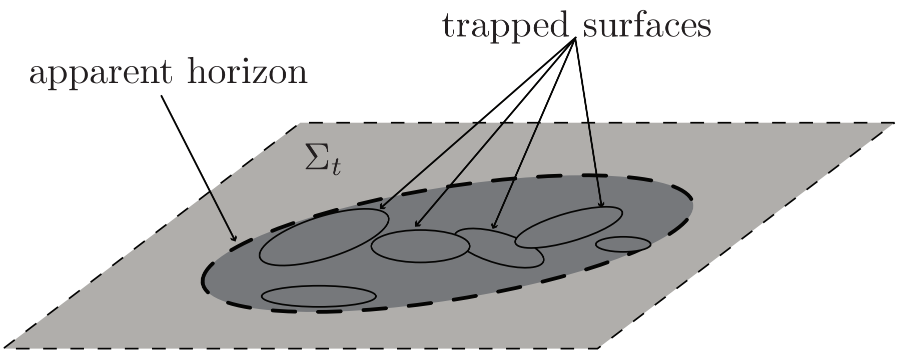


Figure 5.3: An “instant” Σ_t along with some of its trapped surfaces (small black circles), the associated trapped region (dark gray) and the apparent horizon (thick dashed line).

the asymptotically flat Kerr family of (stationary) black hole solutions, the set of all points contained on some trapped surface coincides exactly with the black hole region. Thus it is not unreasonable to consider the existence of trapped surfaces as being the key characterizing feature of a black hole region and this is the basis of the alternative definitions of black objects.

The original such definition was the *apparent horizon*. This begins with the foliation of a spacetime into spacelike hypersurfaces – essentially instants in time. Then at a given instant, the *trapped region* is the union of all the trapped surfaces contained in the hypersurface and the boundary of that region is the *apparent horizon*. It can be shown [97], that on the apparent horizon $\theta_{(\ell)} = 0$ and $\theta_{(n)} < 0$. For the purposes of this Chapter, quasilocal black hole (or brane) horizons in a $(n+1)$-spacetime dimensions will be understood as n-dimensional hypersurfaces (time-evolved apparent horizons) that are foliated by $(n-1)$-dimensional marginally trapped spacelike surfaces (apparent horizons).

The very important feature of quasilocal horizons is that they are not uniquely defined. Given a foliation of spacetime one can define a time-evolved apparent horizon $\triangle$ as the union of the apparent horizons from each surface. Then, it is clear that different foliations will sample different subsets of all the possible trapped surfaces. Thus, different foliations will define different $\triangle$.

5.2.2 The geometry of n-tubes

Time-evolved apparent horizons, event horizons as well as the time-like hypersurfaces of the membrane paradigm are all examples of *n-tubes*, i.e. in $(n+1)$-dimensional spacetime n-dimensional surfaces foliated by $(n-1)$-dimensional spacelike surfaces S_λ. The term “tube” comes from $3+1$ dimensions where the S_λ for horizons are compact objects of topology of S^2 (see Figure 5.4). However, this name will be kept also for black branes where the S_λ are planar and so certainly not compact. Event horizons are n-tubes of null signature which have the correct causal properties as discussed in the previous Section. Time-evolved apparent horizons are $n-$ *tubes* whose S_λ are the apparent horizons found in individual spacetime slices. As will be clear in moment, these are either null (if isolated and in equilibrium) or spacelike (if dynamical and expanding). Finally in the membrane paradigm the $\triangle$ is a timelike surface.

To begin, consider the basic geometry of n-tubes and in particular focus on the spacelike S_λ.

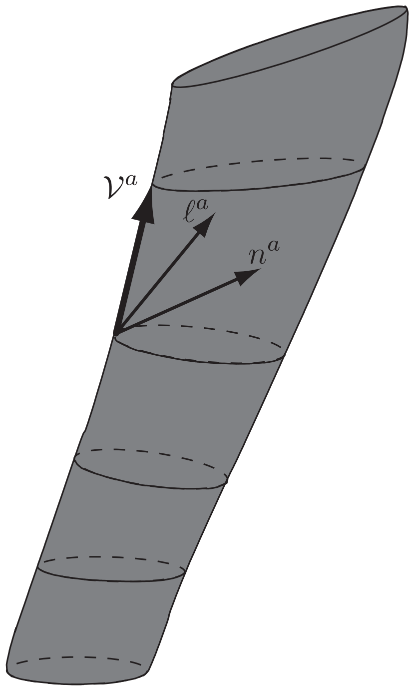


Figure 5.4: A schematic of an n-tube $\triangle$ with compact foliation surfaces S_λ along with the outward and inward pointing null normals to those surfaces. $\mathcal{V}^A$ is the future-pointing tangent to $\triangle$ that is simultaneously normal to the S_λ.

First, the codimension of the S_λ is two and the normal space has Minkowski signature so, as in the previous Section, one can always find null normals ℓ_A and n_A which span that normal space. ℓ^A is taken to be outward-pointing (and so is tangent to $\triangle$ if it is null) while n^A points inwards towards the singularity, see Figure 5.4. The normals are (usually) cross-normalized so that $\ell_A n^A = -1$ which leaves a single scaling degree of freedom in their definition

$$\ell^A \to f\ell^A \text{ and } n^A \to \frac{1}{f}n^A \tag{5.6}$$

for any positive function f.

The next step is to consider how the S_λ fit together to form $\triangle$. In order to make it precise it is natural to introduce the vector field $\mathcal{V}^A$, which evolves the leaves of foliation into each other, i.e. the leaf at λ to the leaf at $\lambda + \mathrm{d}\lambda$ for any λ (see Figure 5.4). Such field is called the evolution vector field and by definition is tangent to time-evolved horizon $\triangle$, as well as normal to each leaf of foliation S_λ. This in turn implies that $\mathcal{V}^A$ is a linear combination of vectors ℓ^A and n^B, here chosen to be

$$\mathcal{V}^A = \ell^A - C\, n^A \tag{5.7}$$

for some function C. If the time-evolved horizon is given by $S = r - r_H\left(x^\mu\right) = \text{const}$[2], then the form normal to it is $\mathrm{d}S$ and the evolution parameter C can be obtained from the condition $\mathrm{d}S\left(\mathcal{V}\right) = 0$.

[2]Note that such parametrization in general might not be the right one, but definitely works if the horizon does not fold over, i.e. r_H is single valued

Clearly thanks to the normalization $\ell_A n^A = -1$, $C > 0 \Leftrightarrow \triangle$ is spacelike, while $C = 0 \Leftrightarrow \triangle$ is null, and $C < 0 \Leftrightarrow \triangle$ is timelike. Further the "time"-rate of change of the area element[3] is

$$\mathcal{L}_{\mathcal{V}}\sqrt{\tilde{q}} = \sqrt{\tilde{q}}\left(\theta_{(\ell)} - C\,\theta_{(n)}\right) . \tag{5.8}$$

Thus, for time-evolved apparent horizons with $\theta_{(\ell)} = 0$ and $\theta_{(n)} < 0$, C also characterizes the evolution of the horizons and is often referred to as the *expansion parameter*. Specifically in such cases

$$\begin{aligned} C<0 \quad &\Leftrightarrow \quad \sqrt{\tilde{q}} \text{ is decreasing} \quad &\Leftrightarrow \quad \mathcal{V}^A \text{ is timelike,}\\ C=0 \quad &\Leftrightarrow \quad \sqrt{\tilde{q}} \text{ is unchanging} \quad &\Leftrightarrow \quad \mathcal{V}^A \text{ is null,}\\ C>0 \quad &\Leftrightarrow \quad \sqrt{\tilde{q}} \text{ is increasing} \quad &\Leftrightarrow \quad \mathcal{V}^A \text{ is spacelike.} \end{aligned} \tag{5.9}$$

In contrast, for event horizons $C = 0$ but away from equilibrium $\theta_{(\ell)} > 0$ (due to the second law [97]) and so the horizon can still expand. For the timelike surfaces of the membrane paradigm none of $\theta_{(\ell)}$, $\theta_{(n)}$ or C vanish. Equation (5.8) will be very important in the following, also in cases when $\theta_{(\ell)}$ is not exactly zero: *it is a universal statement of area law valid for different notions of black hole or black brane horizons.* The freedom of choosing between tubes in bulk, which satisfy (5.8), have a correct equilibrium limit (namely asymptote in the appropriate sense to the event horizon) and satisfy symmetries of boundary theory is conjectured here to capture precisely the freedom in the construction of hydrodynamic entropy current. This implies that demanding (5.8) to be greater or equal to zero is conjectured to be the gravitational counterpart of non-negativity of divergence of hydrodynamic entropy current, i.e.

$$\mathcal{L}_{\mathcal{V}}\sqrt{\tilde{q}} \geq 0 \quad \Leftrightarrow \quad \nabla_\mu J^\mu \geq 0\,, \tag{5.10}$$

or equivalently

$$\theta_{(\ell)} - C\,\theta_{(n)} \geq 0 \quad \Leftrightarrow \quad \nabla_\mu J^\mu \geq 0\,, \tag{5.11}$$

where J^μ is a hydrodynamic entropy current and ∇_μ is the boundary covariant derivative. In the next Section this claim is positively verified in the case of gravity dual to boost-invariant hydrodynamics and in a new article in preparation [108] it is shown to hold as well in the general case of fluid/gravity duality up to second order in gradients. The important lesson, which can be learned from (5.11) is that even beyond equilibrium in two-derivative Einstein gravity the notion of a surface (or evolved surface called here a tube) seems to be sufficient to capture the possible generalizations of entropy of black objects, here suggested by dual gauge theory description. Moreover, if different hydrodynamic entropy currents have different physical meanings in dual gauge theory, then the construction of this Section originally presented in [49] provides a first directly physical interpretation of quasilocal horizons.

5.3 Horizons in the boost-invariant spacetime

5.3.1 Preliminaries

With the tools from the last Section in hand one can now turn to the identification and study of various notions of horizons in the spacetime defined by the bulk metric (4.13) in the hydrodynamic regime. Since the main goal is to study possible notions of entropy in

[3]Quotation marks are used around the word time since if $\mathcal{V}^a$ is spacelike then this is a coordinate rather than physical notion of time.

the dual field theory, the hypersurfaces of interest are those which satisfy the symmetries of the boundary dynamics under consideration (Bjorken flow). This singles out spacelike three-surfaces of constant $\tilde{\tau}$ and r. Such surfaces are *unique* and possess outward and inward pointing null normals

$$\begin{aligned} \ell^A &= \left[\frac{\partial}{\partial \tilde{\tau}}\right]^A + \frac{1}{2} r^2 A(\tilde{\tau}, r) \left[\frac{\partial}{\partial r}\right]^A , \\ n^A &= -\left[\frac{\partial}{\partial r}\right]^A . \end{aligned} \tag{5.12}$$

Those vectors can be obtained (up to overall factor, which is arbitrary provided that both are future oriented) from two ($i = 1, 2$) vectors $V_{(i)}$ of schematic form

$$V_{(i)} = V_{(i)}^{\tilde{\tau}} \partial_{\tilde{\tau}} + V_{(i)}^r \partial_r \tag{5.13}$$

by demanding that both are null and cross-normalized to -1. These are 3 equations for 4 components, which do the job. The remaining unspecified component is precisely the rescaling freedom of $\ell \to f\,\ell$ and $n \to n/f$ and the specific scaling used here is chosen to be consistent with the flow of time at asymptotic infinity.

The hypersurfaces of interest will all lie within the $\tilde{\tau}$ = constant hypersurfaces. Given that coefficients of the metric are expanded in the late time scaling power series (4.14), solutions can be sought in the form

$$r_S(\tilde{\tau}) = \frac{1}{\tilde{\tau}^{1/3}} (r_0 + \frac{1}{\tilde{\tau}^{2/3}} r_1 + \frac{1}{\tilde{\tau}^{4/3}} r_2 + \frac{1}{\tilde{\tau}^2} r_3 + \ldots) . \tag{5.14}$$

The coefficients r_k appearing here will be determined by the conditions imposed, and it turns out that the solutions are unique. Not only is the event horizon uniquely defined, but in this case demanding that the time-evolved apparent horizon shares the symmetries of the spacetime means that one can also select the *unique* time-evolved apparent horizon. In this Section the focus will be on the time-evolved apparent horizon and the event horizon, while in the following Section more general surfaces of the form (5.14) will play an essential role.

5.3.2 The boost-invariant time-evolved apparent horizon

To find marginally trapped three-surfaces within the $\tilde{\tau}$ = constant hypersurfaces, one needs to solve the equation $\theta_{(\ell)} = 0$. Evaluating $\theta_{(\ell)}$ on a hypersurface of the form (5.14) yields

$$\theta_{(\ell)} = \frac{1}{\tilde{\tau}^{1/3}} \left\{ \frac{3}{2} r_0 \left(1 - \frac{\pi^4 \Lambda^4}{r_0^4}\right) + \frac{1}{\tilde{\tau}^{2/3}} \frac{3\pi^4 (3r_1 + 1) \Lambda^4 + 2\pi^3 r_0 \Lambda^3 + r_0^4 (3r_1 + 1)}{2 r_0^4} + \ldots \right\} . \tag{5.15}$$

Solving $\theta_{(\ell)} = 0$ (to third order) shows that there is a unique apparent horizon on the $\tilde{\tau}$ = constant slices at:

$$\begin{aligned} r_{AH}(\tilde{\tau}) &= \frac{1}{\tilde{\tau}^{1/3}} \cdot \left\{ \pi\Lambda - \frac{1}{2} \cdot \frac{1}{\tilde{\tau}^{2/3}} + \left(\frac{1}{9\pi\Lambda} + \frac{1}{24\Lambda} - \frac{\log(2)}{18\pi\Lambda} \right) \cdot \frac{1}{\tilde{\tau}^{4/3}} + \left(\frac{\mathcal{C}}{18\pi^2\Lambda^2} - \frac{25\log(2)}{162\pi^2\Lambda^2} + \right. \right. \\ & \left. \left. - \frac{1}{7776\Lambda^2} + \frac{1}{81\pi^2\Lambda^2} + \frac{11}{432\pi\Lambda^2} - \frac{\log(2)}{24\pi\Lambda^2} + \frac{7\log^2(2)}{162\pi^2\Lambda^2} \right) \cdot \frac{1}{\tilde{\tau}^2} + \ldots \right\}, \end{aligned} \tag{5.16}$$

where $\mathcal{C}$ is Catalan's constant. For the three-surface defined by $r = r_{AH}(\tilde{\tau})$, the inward expansion is

$$\begin{aligned}\theta_{(n)} &= \tilde{\tau}^{1/3}\cdot\left\{-\frac{3}{\pi\Lambda} - \frac{1}{2\pi^2\Lambda^2}\cdot\frac{1}{\tilde{\tau}^{2/3}} - \frac{-1+2\log(2)}{12\pi^3\Lambda^3}\cdot\frac{1}{\tilde{\tau}^{4/3}} + \frac{1}{\tilde{\tau}^2}\left(\frac{7\log^2(2)}{54\pi^4\Lambda^4} - \frac{\log(2)}{24\pi^3\Lambda^4} + \right.\right.\\ &- \left.\left.\frac{25\log(2)}{54\pi^4\Lambda^4} - \frac{1}{2592\pi^2\Lambda^4} + \frac{\mathcal{C}}{6\pi^4\Lambda^4} + \frac{35}{216\pi^4\Lambda^4}\right) + \dots\right\}.\end{aligned} \tag{5.17}$$

It is easy to see numerically that

$$\theta_{(n)} = -\frac{0.95}{\Lambda} - \frac{0.051}{\Lambda^2\tilde{\tau}^{2/3}} - \frac{0.0010}{\Lambda^3\tilde{\tau}^{4/3}} - \frac{0.00039}{\Lambda^4\tilde{\tau}^2} + \dots, \tag{5.18}$$

which is clearly negative and will stay negative in a neighborhood of r_{AH}. Further it is clear from the expression for the outward expansion (5.15) that for $r \approx r_{AH}$, $r > r_{AH} \Rightarrow \theta_{(\ell)} > 0$ and $r < r_{AH} \Rightarrow \theta_{(\ell)} < 0$. That is, there are fully trapped surfaces "just-inside" $r = r_{AH}$ and so this marginally trapped surface bounds a fully trapped region and thus can be identified as a black brane apparent horizon.

The remaining geometric quantities discussed in Section 5.2 can be calculated now. First, requiring that the evolution vector $\mathcal{V}^a$ be tangent to the horizon[4], implies that in the large $\tilde{\tau}$ regime the expansion parameter C has the form

$$C = C_{-1} + \frac{1}{\tilde{\tau}^{2/3}}C_0 + \frac{1}{\tilde{\tau}^{4/3}}C_1 + \frac{1}{\tilde{\tau}^2}C_2 + \frac{1}{\tilde{\tau}^{8/3}}C_3 + \dots \tag{5.19}$$

for some set of coefficients $C_{-1}, \dots C_3$. It is straightforward to see that $C_{-1} = 0$ identically in consequence of the structure of the large $\tilde{\tau}$ expansion (even without using the explicit form of the solution). The coefficients C_0 (coming from zeroth order perfect fluid geometry) and C_1 (first order viscous geometry) also turn out to vanish, so that the leading contribution appears at order $\frac{1}{\tilde{\tau}^2}$. All in all, evaluating (5.19) on time-evolved apparent horizon (so at $r = r_{AH}(\tilde{\tau})$) it is found that

$$C = \frac{1}{9}\cdot\frac{1}{\tilde{\tau}^2} - \left(\frac{\log(2)}{9\pi\Lambda} - \frac{1}{54\Lambda}\right)\cdot\frac{1}{\tilde{\tau}^{8/3}} + \dots \tag{5.20}$$

which is clearly greater than zero. This implies that the horizon is dynamical: spacelike and expanding in area, which can then be cross-checked in two ways. First one can directly calculate the volume element on the three-surfaces being the constant $\tilde{\tau}$ slices of tube of interest

$$\mathrm{vol}_{AH} = \sqrt{\tilde{q}}\,\mathrm{d}y \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \tag{5.21}$$

where, up to third order

$$\begin{aligned}\sqrt{\tilde{q}} &= \pi^3\Lambda^3 - \frac{1}{2}\pi^2\Lambda^2\cdot\frac{1}{\tilde{\tau}^{2/3}} + \left(\frac{1}{4}\pi\Lambda\log(2) + \frac{\pi^2}{24}\Lambda + \frac{\pi}{12}\Lambda\right)\cdot\frac{1}{\tilde{\tau}^{4/3}}\\ &- \left(\frac{5}{216} - \frac{\pi}{144} - \frac{\pi^2}{2592} + \frac{5\log(2)}{216} - \frac{1}{24}\pi\log(2) - \frac{35\log^2(2)}{216}\right)\cdot\frac{1}{\tilde{\tau}^2}.\end{aligned} \tag{5.22}$$

[4] i.e. its action on the form $\mathrm{d}S = \mathrm{d}r - r'_{AH}(\tau)$ gives zero or in other words it is perpendicular to the direction normal to the tube.

Then, to lowest order, the rate of expansion is

$$\frac{1}{\sqrt{\tilde{q}}}\frac{d\sqrt{\tilde{q}}}{d\tilde{\tau}} = \frac{1}{\tilde{\tau}^{5/3}}\left\{\frac{1}{3\pi\Lambda} + \left(-\frac{\log(2)}{3\pi^2\Lambda^2} - \frac{1}{18\pi\Lambda^2} + \frac{1}{18\pi^2\Lambda^2}\right)\cdot\frac{1}{\tilde{\tau}^{2/3}}\right\} \tag{5.23}$$

which is clearly positive. Alternatively using

$$\frac{1}{\sqrt{\tilde{q}}}\frac{d\sqrt{\tilde{q}}}{d\tilde{\tau}} = \frac{1}{\sqrt{\tilde{q}}}\mathcal{L}_{\mathcal{V}}\sqrt{\tilde{q}} = -C\theta_{(n)}\,, \tag{5.24}$$

and substituting in the appropriate values one obtains the same result providing a simple cross-check of the whole approach.

5.3.3 Event horizon

It was emphasized in Section 5.2.1 that the event horizon is somewhat inconvenient to work with, since determining it requires knowing the entire future evolution of the spacetime under consideration. This is indeed the case in the typical situation of computing the evolution of spacetime geometry starting from some initial data. The setting explored in Chapter 4 of this Thesis is in a sense complementary: the spacetime geometry is constructed order by order in a large proper time expansion being a special example of hydrodynamic gradient expansion, starting in the far future at zeroth order. This circumstance makes it possible to determine the location of the event horizon in the late time regime.

The method of finding the event horizon for boost-invariant flow closely resembles the one presented in [69] for the gravity duals to fluid dynamics [10]. The crucial assumption there was that the metric relaxes to (uniformly boosted) AdS-Schwarzschild, where the position of the event horizon is well known. The event horizon for the metrics there was defined as a unique null surface which asymptotically coincides with the event horizon of the static AdS-Schwarzschild dual to the uniform flow at constant temperature. Despite the fact that this is not the case for boost-invariant flow, it is still possible to find a unique null surface which is interpreted as the event horizon. The key observation is that the event horizon should coincide with the time-evolved apparent horizon in the large-proper time regime and within the scaling limit. Its radial position in AdS should depend on proper time only, which reflects the boost-invariance (no rapidity dependence) together with translational and rotational symmetry in the perpendicular directions (no $\vec{x}_\perp$ dependence). If r is the radial direction in AdS space, $\tilde{\tau}$ the proper time and $r_{EH}(\tilde{\tau})$ expresses the time evolution of the horizon, then the equation defining the sought codimension-one surface in AdS takes the form

$$r - r_{EH}(\tilde{\tau}) = 0. \tag{5.25}$$

The covector normal to the surface is $dr - r'_{EH}(\tilde{\tau})\,d\tilde{\tau}$ and requiring it is null gives the equation for $r_{EH}(\tilde{\tau})$

$$A(\tilde{\tau}, r_{EH})\cdot r_{EH}^2 - 2r'_{EH} = 0, \tag{5.26}$$

where for clarity the dependence of r_{EH} on $\tilde{\tau}$ is omitted. This equation can be solved perturbatively in the scaling limit. Using the late time solution valid up to third order in the late proper time expansion one finds

$$
\begin{aligned}
r_{EH} \;=\; & \frac{1}{\tilde{\tau}^{1/3}}\Bigg\{\pi\Lambda - \frac{1}{2}\cdot\frac{1}{\tilde{\tau}^{2/3}} + \left(\frac{1}{6\pi\Lambda} - \frac{1}{24\Lambda} - \frac{\log(2)}{18\pi\Lambda}\right)\cdot\frac{1}{\tilde{\tau}^{4/3}} + \left(\frac{\mathcal{C}}{18\pi^2\Lambda^2} - \frac{5}{324\pi^2\Lambda^2} + \right.\\
& \left. -\frac{1}{7776\Lambda^2} + \frac{7}{432\pi\Lambda^2} - \frac{17\log(2)}{81\pi^2\Lambda^2} - \frac{\log(2)}{24\pi\Lambda^2} + \frac{7\log^2(2)}{162\pi^2\Lambda^2}\right)\cdot\frac{1}{\tilde{\tau}^2}\Bigg\}. \qquad (5.27)
\end{aligned}
$$

Comparing (5.16) with (5.27) it turns out that the time-evolved apparent horizon coincides with the event horizon in the leading and first subleading orders, which is in agreement with the observation that C, the expansion parameter on time-evolved apparent horizon, is non-zero only in the second and higher orders in $1/\tilde{\tau}^{2/3}$ expansion. The second orders differ and the time-evolved apparent horizon becomes spatial. It is easy to see that time-evolved apparent horizon indeed lies inside the event horizon at given instant of $\tilde{\tau}$, since the difference between $r_{EH} - r_{AH}$

$$
r_{EH} - r_{AH} = \frac{1}{18\pi\Lambda}\cdot\frac{1}{\tilde{\tau}^{5/3}} + \ldots \qquad (5.28)
$$

is positive in the leading (here being second) order. The fact that event and time-evolved apparent horizons match in the perfect fluid and viscous orders raises a question whether this is a coincidence or there is some underlying principle. The answer to this question was provided for the first time in [49] and is related to Weyl covariance of boundary hydrodynamics in those orders of gradient expansion, as explained in the next Section.

Finally, observe that in the naive limit $\tilde{\tau} \to \infty$ the boost-invariant metric relaxes to the empty AdS_5 metric instead of the static AdS-Schwarzschild solution. However, this is not so strange from the dual CFT point of view, where the fluid is expanding to infinity and its energy density becomes smaller and smaller. It means that the boundary system does not permanently thermalize to non-zero temperature. The interesting feature of the boost-invariant flow is an apparent thermalization, which expresses itself as an applicability of the equations of hydrodynamics in the late stages of the evolution.

5.3.4 Revisiting the scaling limit

The symmetries of boost-invariant flow make it possible to seek the location of the event horizon considering only the variables r and $\tilde{\tau}$. It is possible then to focus only on the $\mathrm{d}r - \mathrm{d}\tilde{\tau}$ part of the full metric, which at leading order takes the form

$$
\mathrm{d}s^2 = 2\mathrm{d}\tilde{\tau}\,\mathrm{d}r - r^2\left\{1 - \frac{\pi^4\Lambda^4}{(r\,\tilde{\tau}^{1/3})^4}\right\}\mathrm{d}\,\tilde{\tau}^2 + \ldots \qquad (5.29)
$$

The scaling limit discussed at length in Chapter 4 involved introducing the scaling variable $\tilde{v} \sim r\,\tilde{\tau}^{1/3}$ which is kept fixed as $\tilde{\tau} \to \infty$. This motivates the following change of variables

$$
\begin{aligned}
\tilde{\tau} &= \left(\frac{2u}{3}\right)^{3/2},\\
r &= \sqrt{\frac{3}{2u}}\tilde{v}, \qquad (5.30)
\end{aligned}
$$

which leads to

$$
ds^2 = 2\mathrm{d}u\,\mathrm{d}\tilde{v} - \tilde{v}^2\left\{1 - \frac{\pi^4\Lambda^4}{\tilde{v}^4}\right\}\mathrm{d}u^2 + \cdots + O\left(\frac{1}{u}\right) \qquad (5.31)
$$

showing that this part of the metric takes precisely the same form as the corresponding part of the static black brane metric, with $\tilde{v}$ denoting the radial coordinate and u Eddington-Finkelstein ingoing time coordinate. This implies that at leading order in the late-time expansion the problem of determining radial geodesics[5] in the asymptotic boost-invariant geometry is the same as in the static case. It is then not surprising that the naive position of the horizon coincides asymptotically with the actual event horizon. Note however that these considerations do not imply that the asymptotic geometry is static. Clearly, the remaining terms in the metric are time-dependent after this coordinate transformation, even though the area of the event horizon remains constant in that order.

5.4 Phenomenological notions of entropy

5.4.1 Introduction

The equilibrium states of black objects are thermodynamic in nature. Their entropy is associated with the area of spacelike slices of the event horizon in an unambiguous way and the second law of thermodynamics is linked with area theorem. The property that the area of the event horizon is non-decreasing continues to hold in a generic dynamical setting. This prompts the question whether there is a sensible notion of entropy valid in such a non-equilibrium situation. However, as anticipated in previous Sections, there are hypersurfaces of non-decreasing area other than the event horizon (which coincide with it in the static case). The notion of entropy thus becomes less clear in these cases, as frequently discussed in the literature. The AdS/CFT correspondence makes it possible to view this problem from the gauge theory perspective. As anticipated in Section 5.2.1, the teleological nature of the event horizon leads to acausal behavior of gauge theory entropy associated with it [21, 90]. Although the bulk description is under control precisely when the field theory is strongly coupled, which in itself makes it hard to analyze directly, in the near-equilibrium regime one can base on intuition following from hydrodynamic considerations on gauge theory side of duality [12, 10, 13].

5.4.2 Entropy from gravity

As reviewed in Section 5.2, in a dynamical setting it is no longer clear if there is an appropriate geometrical notion which can be used for the definition of entropy. For example both time-evolved apparent and event horizons appear to give rise to notions of entropy, which satisfy the second law of thermodynamics and coincide in equilibrium. This provides motivation to look more generally at the dynamics of hypersurfaces whose area is non-decreasing. The starting point should be equation (5.8). This formula determines the rate of change of the area element of a general 3-hypersurface in the terms of the expansions $\theta_{(n)}$, $\theta_{(\ell)}$ and the expansion parameter C. In the boost-invariant case the 3-hypersurfaces consistent with the boundary symmetry have the general form (5.14). Their area is given by

$$\mathcal{A} = \pi^3 \Lambda^3 \left\{ 1 + \frac{3r_1 + 1}{\pi\,\Lambda} \cdot \frac{1}{\tilde{\tau}^{2/3}} + \frac{36 r_1^2 + 24 r_1 + 36\pi r_2 + 2\pi + 5\log(2)}{12\pi^2\,\Lambda^2} \cdot \frac{1}{\tilde{\tau}^{4/3}} \right\}, \tag{5.32}$$

[5] As stressed previously, this is all that is needed to determine the location of the event horizon.

where r_i come from (5.14), leading to the following form of "entropy" defined by such hypersurfaces[6]

$$S = \frac{N_c^2}{2\pi}\mathcal{A}. \tag{5.33}$$

Equation (5.8), which expresses the change of area of the hypersurface sections becomes now

$$\begin{aligned}\mathcal{L}_\mathcal{V}\sqrt{\tilde{q}} &= \sqrt{\tilde{q}}(\theta_{(\ell)} - C\theta_{(n)}) = \\ &= -\frac{6r_1+2}{3\pi\Lambda}\cdot\frac{1}{\tilde{\tau}^{5/3}} + \frac{18r_1^2+12r_1-2\pi(18r_2+1)+6-5\log(2)}{9\pi^2\Lambda^2}\cdot\frac{1}{\tilde{\tau}^{7/3}}\end{aligned} \tag{5.34}$$

with r_0 set to $\pi\Lambda$ to match the thermodynamic entropy when all gradient corrections are discarded. For the leading term (at order $1/\tilde{\tau}^{5/3}$) to be non-negative one gets the bound $r_1 < -1/3$, and then requiring that the following term be smaller gives an allowed range for r_2. At this level of analysis this is all one gets; r_1 is not fixed. In particular, note that there are surfaces outside the event horizon which are acceptable from this point of view. Both the entropy via the event or time-evolved apparent horizon provide unique $r_1 = -\frac{1}{2}$, which lies in the allowed range.

The entropy density[7] obtained from the third order expression for the event horizon reads

$$\begin{aligned}s_{EH} = \frac{1}{2}N_c^2\pi^2\Lambda^3\frac{1}{\tilde{\tau}}\Bigg\{1 - \frac{1}{2\pi\Lambda}\cdot\frac{1}{\tilde{\tau}^{2/3}} + \left(\frac{1}{4\pi^2\Lambda^2} + \frac{1}{24\pi\Lambda^2} + \frac{\log(2)}{4\pi^2\Lambda^2}\right)\cdot\frac{1}{\tilde{\tau}^{4/3}} + \\ -\left(\frac{35\log^2(2)}{216\pi^3\Lambda^3} + \frac{\log(2)}{24\pi^2\Lambda^3} + \frac{31\log(2)}{216\pi^3\Lambda^3} + \frac{1}{2592\pi\Lambda^3} + \frac{5}{144\pi^2\Lambda^3} + \frac{35}{216\pi^3\Lambda^3}\right)\cdot\frac{1}{\tilde{\tau}^2}\Bigg\},\end{aligned} \tag{5.35}$$

whereas for the time-evolved apparent horizon it takes the form

$$\begin{aligned}s_{AH} = \frac{1}{2}N_c^2\pi^2\Lambda^3\frac{1}{\tilde{\tau}}\Bigg\{1 - \frac{1}{2\pi\Lambda}\cdot\frac{1}{\tilde{\tau}^{2/3}} + \left(\frac{1}{12\pi^2\Lambda^2} + \frac{1}{24\pi\Lambda^2} + \frac{\log(2)}{4\pi^2\Lambda^2}\right)\cdot\frac{1}{\tilde{\tau}^{4/3}} + \\ -\left(\frac{35\log^2(2)}{216\pi^3\Lambda^3} - \frac{\log(2)}{24\pi^2\Lambda^3} + \frac{5\log(2)}{216\pi^3\Lambda^3} - \frac{1}{2592\pi\Lambda^3} - \frac{1}{144\pi^2\Lambda^3} - \frac{5}{216\pi^3\Lambda^3}\right)\cdot\frac{1}{\tilde{\tau}^2}\Bigg\}.\end{aligned} \tag{5.36}$$

Numerically one finds

$$s_{AH} = \frac{1}{2}N_c^2\pi^2\Lambda^3\frac{1}{\tilde{\tau}}\left\{1 - \frac{0.16}{\Lambda}\cdot\frac{1}{\tilde{\tau}^{2/3}} + \frac{0.039}{\Lambda^2}\cdot\frac{1}{\tilde{\tau}^{4/3}} - \frac{0.0065}{\Lambda^3}\cdot\frac{1}{\tilde{\tau}^2}\right\} \tag{5.37}$$

and for the event horizon

$$s_{EH} = \frac{1}{2}N_c^2\pi^2\Lambda^3\frac{1}{\tilde{\tau}}\left\{1 - \frac{0.16}{\Lambda}\cdot\frac{1}{\tilde{\tau}^{2/3}} + \frac{0.056}{\Lambda^2}\cdot\frac{1}{\tilde{\tau}^{4/3}} - \frac{0.018}{\Lambda^3}\cdot\frac{1}{\tilde{\tau}^2}\right\}. \tag{5.38}$$

The key observation already mentioned before is that the event and time-evolved apparent horizons coincide at the leading and first subleading orders, which is a hint that also in the case of a general surface there should be no ambiguity until the second subleading order. If one is to identify the entropy defined here with the field theory observable, as required by the

[6] Note that in the units used here (AdS radius set to 1) $G_N^{-1} = 2\pi^{-1}N_c^2$.

[7] Entropy density is understood as entropy per unit volume, which in the proper time – rapidity coordinates involves a factor of $\tilde{\tau}$.

AdS/CFT correspondence, then it should be Weyl-covariant in the boundary sense. To do this explicitly one would need to solve the Einstein equations with the boundary metric given by

$$ds_4^2 = e^{-2\omega(\tau)}\left\{-d\tau^2 + \tau^2 dy^2 + dx_\perp^2\right\} \tag{5.39}$$

where $\omega(\tau)$ is a conformal factor having the form of an expansion in powers of $\frac{1}{\tau^{2/3}}$. The entropy computed this way would be Weyl-covariant (i.e. proportional to the appropriate power of the conformal factor) only for $r_1 = -1/2.$, i.e. the value assumed by r_1 in the case of the event or time-evolved apparent horizon (which coincide at this order). The quick way to get this answer is to write the entropy (5.33) in terms of temperature and velocity, whose transformation rules under Weyl rescalings are known. This procedure parallels the field theory analysis reviewed earlier. The first step is to factor out the thermodynamic entropy which sets the Weyl transformation property of the entropy density. This leads to

$$s = \frac{1}{2}N_c^2\pi^2 T(\tilde{\tau})^3\Bigg\{1 + \frac{6r_1+3}{2\pi\Lambda}\cdot\frac{1}{\tilde{\tau}^{2/3}} + \tag{5.40}$$

$$+\frac{36r_1^2+42r_1+36\pi r_2+2\pi+9+4\log(2)}{12\pi^2\Lambda^2}\cdot\frac{1}{\tilde{\tau}^{4/3}}\Bigg\}. \tag{5.41}$$

Since there are no Weyl-covariant scalars nor vectors at first order in derivatives, the only way that this formula can be Weyl-covariant is if the first subleading term vanishes, which determines $r_1 = -1/2$. When this result is substituted into the equation (5.34), one finds

$$\begin{aligned}\mathcal{L}_\mathcal{V}\sqrt{\tilde{q}} &= \sqrt{\tilde{q}}(\theta_{(\ell)} - C\theta_{(n)}) = \\ &= \frac{1}{3\pi\Lambda}\cdot\frac{1}{\tilde{\tau}^{5/3}} + \frac{9-4\pi(18r_2+1)-10\log(2)}{18\pi^2\Lambda^2}\cdot\frac{1}{\tilde{\tau}^{7/3}}.\end{aligned} \tag{5.42}$$

The leading contribution ensures positive entropy production due to the shear viscosity so there are no further constraints on r_2. The appearance of possible freedom in choosing r_2 can be understood following again the hydrodynamic argument presented in Chapter 2. In the first place, note that the formula (5.42) is evaluated using the bulk Eddington-Finkelstein proper time $\tilde{\tau}$, which raises the question about its relation with boundary proper time coordinate τ. Such a mapping freedom has been addressed using Weyl-covariant language in [69] and amounts to the trivial mapping in the first two orders of the gradient expansion, with an ambiguity showing up at the second order. In the case of boost-invariant flow the most general mapping (up to second order) takes the form

$$\tilde{\tau} \longrightarrow \tilde{\tau}\left(1 + \frac{\delta A_1}{\Lambda^2\,\tilde{\tau}^{4/3}}\right), \tag{5.43}$$

where δA_1 is a constant parameter multiplying Weyl-covariant scalar S_1. For such mappings to make sense within the context of the gradient expansion this parameter must be suitably bounded as explained earlier. The rest of the ambiguity can be understood following the Weyl analysis of the properties of gravitational entropy. At second order in gradients there are two Weyl-covariant scalars (S_1 and S_3) which are non-trivial when evaluated on the boost-invariant solution. Since mapping freedom is identified (partly) with the S_1 contribution, it is clear the r_2 must come from the relevant combination of S_1 and S_3[8]. Evaluating (5.40) with $r_1 = -1/2$ gives

$$s = \frac{1}{2}N_c^2\pi^2 T(\tilde{\tau})^3\left\{1 + \frac{-3+36\pi r_2+2\pi+4\log(2)}{12\pi^2\tilde{\tau}^{4/3}\Lambda^2}\right\} \tag{5.44}$$

[8]Note that there is no contribution from Weyl-covariant vectors, since those are transverse to velocity.

Comparing this with (3.17) one finds

$$r_2 = \frac{2}{9\pi}\left(A_1 - A_3 + B_1\right) + \frac{1}{12\pi} - \frac{1}{18} - \frac{\log(2)}{9\pi} \tag{5.45}$$

Although this is all one can get from the analysis of the gravity dual to the boost-invariant flow[9] it is reassuring that at least in this case the gravity picture is capable of capturing the ambiguities of the boundary phenomenological construction.

5.4.3 A phenomenological definition of black brane entropy

The freedom in the definition of the hydrodynamic entropy current on the gravity side follows not from the various possible notions of horizon, but rather from adopting the phenomenological construction in the bulk, which is analogous to the boundary one. The surfaces considered are not horizons in any of the usual senses, but they do have the property that their area increases, they obey all symmetries of dual theory and have a correct equilibrium limit. The last requirement can be made precise by adopting the framework of slowly evolving horizons, for details of this construction with application to holographic boost-invariant hydrodynamics see original publication [49]. In order to understand the phenomenological definition of black brane entropy in greater generality one would of course need to go beyond the Bjorken flow example and consider the equation (5.8) evaluated on the gravity dual to the general hydrodynamics. This is possible employing the Weyl-covariant formulation in the bulk and is a subject of ongoing work [108].

5.5 Conclusions and outlook

The main goal of this Chapter based on original publication [49] was to explore the relationship between the notions of entropy on both sides of AdS/CFT duality. This led to a phenomenological definition of black brane entropy on the gravity side, which was inspired by the corresponding construction in hydrodynamics. In the case of Bjorken flow the freedom inherent in this definition accounts for the entire ambiguity appearing in the hydrodynamic entropy current in this case. This led to an understanding why the event horizon coincides in the leading and first subleading orders of the gradient expansion with the unique time-evolved apparent horizon compatible with the boundary flow. The origin of this circumstance is the Weyl covariance of the boundary hydrodynamics at those orders of gradient expansion (note that conformal anomaly enters the expansion at order 4 in $3+1$ dimensions). Generalization of this argument to arbitrary flow within fluid/gravity duality of [35] is a subject of ongoing research [108].

It is natural to ask what is the physical relevance of the potential ambiguity in the definition of entropy current. In the case of local entropy production such an ambiguity might signal lack of physical meaning. This however should not be disturbing, because the thermodynamic notion of entropy makes sense only in equilibrium. Since one expects that systems described by hydrodynamics equilibrate due to dissipative effects, the total entropy can be calculated in the late stages of evolution and is given by the thermodynamic entropy. On the gravity side

[9] Note that in hydrodynamics due to the requirement of positive divergence of the entropy current, A_1 and B_1 contributions to the entropy current are not independent, but rather linked by the equation (2.25). However such considerations were done under the assumption that the shear tensor vanishes locally, which is never the case in the Bjorken picture unless all the dissipative contributions are negligible.

this translates to the notion of isolated horizons (see [109] for a review) as those for which entropy can be defined precisely. On the other hand, it might still be the case that different surfaces in the bulk carry information about different physical characteristics of dual gauge theory. Further investigations of that issue surely deserve further work.

Apart from some more or less obvious generalizations it would be interesting to explore these ideas in the context of equilibration of the boundary quantum field theory perturbed out of equilibrium by localized sources (in the spirit of [21, 67, 45]). In the case of planar horizons considered here there can be widely separated regions, of which some are in local equilibrium while others are not.

This work also sheds some light on the long-standing discussion as to whether it is more “correct” to consider event or (time-evolved) apparent horizons in physical situations. Instead of there being just two choices, it is now proposed that are many more and the uncertainty as to which one should considered actually reflects a physical ambiguity in the proper definition of entropy and other quantities such as energy as one moves away from equilibrium. Again these issues deserve further investigation.

Chapter 6

Higher derivative corrections to gravity action and hydrodynamics

6.1 Motivation

The studies of universal gravity action consisting of Einstein-Hilbert term and negative cosmological constant have led to many fruitful insights about dynamics of non-Abelian gauge theories at strong coupling. The universality of such description lies in the fact that it is the same for any $(3+1)$-dimensional hCFT in planar strongly coupled regime, no matter what is its matter content, the gauge group, whether it is supersymmetric or not provided the only operator with nonzero expectation value is the energy-momentum tensor. In the gravitational language this amounts to the fact that the universal gravity action is a *consistent* truncation of 10-dimensional dynamics to 5 dimensions. The holographic result of biggest impact delivered by those studies is definitely the celebrated ratio of shear viscosity to entropy density equal to $1/4\pi$ for $\mathcal{N}=4$ SYM in planar strongly coupled regime [30][1]. Subsequent developments revealed that η/s is necessarily $1/4\pi$ in *all* (in any dimension $(d+1)>(1+1)$, conformal or not, supersymmetric or non-supersymmetric) holographic gauge theories, as long as the dual gravity action is based on two-derivative lagrangian [62, 63]. As anticipated before, Kovtun, Son and Starinets influenced by early results [30, 62] and comparisons of transport properties of holographic and real-world fluids proposed in [11] that $\eta/s = 1/4\pi$ might be a universal lower bound for the ratio of those quantities for *any* fluid (the KSS bound), which triggered a lot of interest in holographic hydrodynamics. It is worth noting that experimental estimates of η/s for nuclear matter at RHIC are of order of $1/4\pi$ [84, 1, 110].

In order to get closer with holographic models to real-world non-Abelian gauge theory of interest – the QCD – it is needed to go beyond the paradigm of universal gravity action. The first route is to break conformal symmetry in the holographic context, which is an active area of research by itself, but not covered in this Thesis (see e.g. [111, 112, 113, 114]). The second interesting direction, being the subject of this Chapter, is to include higher derivative terms in the dual 5-dimensional effective gravity action[2]. By doing this one extends the range of physical characteristics of dual gauge theories away from planar limit at *infinite* coupling. In particular, the canonical example of such expression being a certain contraction pattern

[1]From nowadays perspective it is clear that the result of those authors is valid for any hCFT in $(3+1)$ dimensions, since it is derived from universal gravity action.

[2]Of course, as a next step in this program one would like to both break the conformal symmetry and include higher derivative terms in the dual gravity action.

of four Weyl tensors appearing in type IIB string theory [115] in the case of superconformal holographic gauge theories (e.g. $\mathcal{N} = 4$ SYM) is interpreted as a correction to their dynamics from *finite* value of 't Hooft coupling [36]. That is, by studying the solutions of equations of motion for the action consisting of Einstein-Hilbert term, negative cosmological constant and this particular contraction pattern of Weyl tensors one can calculate various properties of those planar hCFTs for large, but in that case *finite* coupling [36]. In particular, such a correction is known to increase the ratio of shear viscosity to entropy density [116, 117, 118] and in the context of the conjectured KSS bound it was very interesting to understand whether there might be another corrections, which for some hCFT(s) would lead to bound violation. Early result by Kats and Petrov [119] (see also [120]), confirmed later by a detailed analysis in [19], implies that for certain hCFTs with the dual gravity action having curvature squared interaction there is a mild violation of the KSS bound. The question whether there exists another bound on the ratio of shear viscosity to entropy density is unsettled, it may well be that η/s can be driven consistently towards 0 (see [110] for an interesting discussion on these topics).

Higher derivative terms usually lead to tachyons unless treated perturbatively on top of two-derivative lagrangian (see however [120, 121, 122, 123, 124, 125, 126, 127]). Generally one expects a whole bunch of higher derivative terms for the metric with non-trivial couplings to other bulk (in particular scalar) fields, as well as higher derivative terms for those fields. The self-consistency of the approach implies that inclusion of higher derivative expressions needs to be understood as a bulk gradient expansion with microscopic scale set *ideally* by something of order of Planck length (or at least parametrically smaller than the curvature scale of AdS) and macroscopic one being the curvature scale of AdS geometry [19]. Note also that once going beyond supergravity approximation the graviton sector might not be a consistent truncation of 10-dimensional physics, or in other words in the dual hCFT language other operators than the energy-momentum tensor might acquire an expectation value.

In this Chapter, based on [64], the focus is on the graviton sector in the absence of any additional bulk matter fields and in the 5-dimensional effective action all *independent* higher derivative terms quadratic and cubic in curvature are included. Furthermore two out of five possible quartic terms are taken as a representative of a more general case [128]. Such an effective gravitational action is used then to provide the corrections to thermodynamic and transport properties of a general class of hCFTs from leading higher derivative terms coming from the graviton sector with gravitational coupling constants re-interpreted in terms of gauge theory parameters. These results might be used for comparisons with QCD plasma (see [36, 19, 129] for interesting proposals), provided the bulk gravitational construction is self-consistent. In particular, complications arising from the presence of light (of masses of order of inverse AdS radius) 5-dimensional scalar fields coupled linearly to higher derivative terms are outlined, as well as problems coming from exactly massless scalars in the effective gravitational action.

6.2 A general framework for studying higher derivative corrections to gravity action

The universal gravity action consisting of Einstein-Hilbert term and negative cosmological constant comes from Kaluza-Klein reduction of supergravity on compact 5-dimensional manifold with only a single zero mode kept nontrivial being a five-form field. Going beyond two-derivative description in 5 dimensions is non-trivial, since the full geometry is 10-dimensional and some fields coming from Kaluza-Klein reduction may couple to terms higher order in curvature. This Chapter focuses on thermo- and hydrodynamics in the absence of conserved charges, so $U(1)$ bulk gauge fields are set to zero by default (for an extensive discussion of the effect of higher derivative corrections on hydrodynamics with conserved charges see [75][3]). The case of massless scalar field is discussed below and furthermore it is assumed here and explained later in the text that there are not any light (of masses of inverse curvature radius of AdS) scalars coupled linearly to higher curvature terms. With these assumptions one is led to construct the gravity action from higher curvature terms order by order in number of derivatives as required by the principle of gradient expansion. It is worth noting at this point that including higher curvature terms does not destroy AdS vacuum, which is implied by the conformal invariance of dual gauge theory, but it may change its radius of curvature with respect to two-derivative value. The coefficients at each term in the gradient-expanded gravity action should be *typically* suppressed by required ratio of Planck length to a curvature scale of AdS, which as will be shown below leads to a hierarchy between these parameters. Nontrivial situations may develop if there is no such suppression, which was the reason for additional assumptions about scalar fields in the gravitational solutions.

Consider now the simplest case of including terms quadratic in curvature in the 5-dimensional gravity action. In total there are 3 such terms leading to an expression of the form

$$I_{gravity} = \frac{1}{2l_P^3}\int_M \mathrm{d}^5x \left\{\mathcal{R} + \frac{12}{\mathcal{L}^2} + \mathcal{L}^2\left(\gamma_1 \mathcal{R}_{ABCD}\mathcal{R}^{ABCD} + \gamma_2 \mathcal{R}_{AB}\mathcal{R}^{AB} + \gamma_3 \mathcal{R}^2\right) + \ldots\right\}, \quad (6.1)$$

where "…" denotes terms yet higher orders in derivatives and γ_i are dimensionless parameters, each of order of $l_P^2/\mathcal{L}^2$. Analogously to the two-derivative case discussed in Chapter 1, the action (6.1) needs to be supplemented with suitable boundary and counter terms, which will be discussed later in the text. The crucial aspect of (6.1) is that equations of motion need to be solved perturbatively on top of two-derivative solution, which implies that one can perform *freely* a field (so here the metric) redefinitions in (6.1) to change the form of the higher derivative correction with $\mathrm{O}(l_P^4/\mathcal{L}^4)$ precision in this case. This leads to (6.1) written *equivalently* as

$$I_{gravity} = \frac{1}{2l_P^3}\int_M \mathrm{d}^5x \left\{\mathcal{R} + \frac{12}{\mathcal{L}^2} + \mathcal{L}^2 \alpha_1 \mathcal{C}_{ABCD}\mathcal{C}^{ABCD} + \ldots\right\}, \quad (6.2)$$

where $\mathcal{C}$ is a 5-dimensional Weyl tensor, which by definition vanishes in AdS vacuum and "…" again denotes higher derivative corrections suppressed in that order of derivative expansion. It can be checked further, that most of cubic terms can be removed by another field redefinition, so that only a single term being a simple contraction of Weyl tensor is left

$$I_{gravity} = \frac{1}{2l_P^3}\int_M \mathrm{d}^5x \left\{\mathcal{R} + \frac{12}{\mathcal{L}^2} + \mathcal{L}^2\, \alpha_1\, \mathcal{C}_{ABCD}\mathcal{C}^{ABCD} + \mathcal{L}^4\, \alpha_2\, \mathcal{C}_{AB}{}^{CD}\mathcal{C}_{CD}{}^{EF}\mathcal{C}_{EF}{}^{AB} + \ldots\right\}. \quad (6.3)$$

It is important to stress that the action (6.3) is the most general one in the absence of scalar fields in the perturbative approach to higher derivative terms. Further analysis in [128] reveals that there are 5 independent contributions from terms quartic in curvature, but here only the following contraction pattern appearing naturally in type IIB string theory [115] is taken into account

[3]More correctly, this is hydrodynamics with an anomalous current in the light of [14].

$$W\left(\mathcal{C}\right)=\mathcal{C}_{ABCD}\,\mathcal{C}^{EBCF}\,\mathcal{C}^{AGH}{}_{E}\,\mathcal{C}^{D}{}_{GHF}-\frac{1}{4}\mathcal{C}_{ABCD}\,\mathcal{C}^{AB}{}_{EF}\,\mathcal{C}^{CE}{}_{GH}\,\mathcal{C}^{DFGH} \tag{6.4}$$

as a representative of a general case. The final form of the gravity action is thus

$$\begin{aligned} I_{gravity} = \quad & \frac{1}{2l_P^3}\int_M \mathrm{d}^5x\Big\{\mathcal{R}+\frac{12}{\mathcal{L}^2}+\mathcal{L}^2\,\alpha_1\,\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}+ \\ & \mathcal{L}^4\,\alpha_2\,\mathcal{C}_{AB}{}^{CD}\mathcal{C}_{CD}{}^{EF}\mathcal{C}_{EF}{}^{AB}+L^6\,\alpha_3\,W\left(C\right)+\ldots\Big\}. \end{aligned} \tag{6.5}$$

Note that the gradient expansion *ideally*[4] implies that $\alpha_i = \mathcal{O}\left(l_P^{2i}/\mathcal{L}^{2i}\right)$, forcing a hierarchy between the couplings

$$\alpha_1 \gg \alpha_2 \gg \alpha_3. \tag{6.6}$$

The equations of motion need to be solved perturbatively in couplings α_i on top of solution of two-derivative equations of motion. If the light 5-dimensional scalar fields can be set consistently to 0 in the 5-dimensional effective action, it is sufficient to keep only first two higher derivative terms and solve equations of motion to quadratic order in α_1 and linear in α_2, so that the overall order of this perturbative expansion is $\mathrm{O}(l_P^4/\mathcal{L}^4)$. If the scalar is linearly coupled to one of higher derivative interaction it will acquire non-trivial profile and integrating it out spoils the hierarchy of terms (6.6) as will be shown in Section 6.5. It is assumed here that no such field is present in the system. The things are more subtle in the case of exactly massless scalar, but it turns out that in such cases typically $\alpha_2 = 0$ [64], so that one is left only with quadratic and quartic contribution. Because the hierarchy of terms (6.6) cannot be now trusted (see Section 6.5 for explanation), the best one can hope for is to solve equations linearly both in α_1 and α_3[5]. It needs to be stressed again that quartic contribution is not the most general one possible and a more thorough studies are required at that order [128].

6.3 Solving equations of motion

As stressed before, Einstein's equations for the action (6.5) need to be solved perturbatively starting with supergravity solution, e.g. black brane metric (1.26) or gravity dual to boost-invariant flow (4.1). Coefficients appearing in higher derivative expansion of the bulk action have direct physical interpretation in terms of parameters of dual hCFTs. In particular, coefficient in front of the particular term quartic in Weyl tensor (6.4) is interpreted in many cases (including $\mathcal{N}=4$ SYM) as a leading correction from finite value of 't Hooft coupling (see [36] for a very extensive discussion on this topic). The physical meaning of other coefficients is explained in the next Section. What needs to be stressed is that higher derivative corrections are not going to change the *form* of hydrodynamic expansion, e.g. in the case of Bjorken flow large-τ scaling of the energy density is going to remain $-4/3$ also in the presence of higher derivative terms in the dual gravitational description. This is because hydrodynamic behavior is universal: the only underlying assumption is large separation of scales between microscopic dynamics and scales of changes of macroscopic quantities. What is going to change on the other hand are coefficients appearing in thermodynamic quantities, i.e. the overall scaling of

[4] Confront it with assumptions about the absence of light scalar fields in the gravitational solution.

[5] It is also possible that more involved situations develop, so that light scalars couple to higher derivative terms in such a way, that their presence does not matter up to desired order.

energy density with respect to temperature, as well as values of transport coefficients, e.g. the ratio of shear viscosity to entropy density is no longer going to be $1/4\pi$. Note that since higher derivative corrections do not destroy vacuum AdS solution, no new hydrodynamic coefficients excluded by conformal invariance (the lowest being the bulk viscosity [111, 112]) are going to appear. Note also that standard relations between thermodynamic quantities forced by conformal invariance are also going to remain valid in such case, i.e. $s = 4/3\,\epsilon\,T^{-1}$ in the absence of conserved charges.

In order to solve gravitational equations of motion in the presence of higher derivative interactions it is most convenient to adopt an effective action method (reviewed for example in [117]) rather than write full Einstein's equations for general metric and only then seek solutions. For illustrative purposes consider now the gravity dual to boost-invariant flow, with any other situation being just a straightforward generalization of that case. The most general form of the metric Ansatz with all symmetries of the boundary dynamics (here being the boost-invariant flow) taken into account reads

$$\mathrm{d}s^2 = 2\,G_{\tau z}\left(\tau, z\right)\mathrm{d}\tau\mathrm{d}z + G_{zz}\left(\tau, z\right)\mathrm{d}z^2 + \frac{1}{z^2}\left\{-e^{a(\tau,z)}\mathrm{d}\tau^2 + \tau^2 e^{b(\tau,z)}\mathrm{d}y^2 + e^{c(\tau,z)}\mathrm{d}\mathbf{x}_\perp^2\right\}. \quad (6.7)$$

In this expression G's are gauge degrees of freedom on gravity side related to diffeomorphisms in τ and z. Choosing $G_{\tau z} = 0$ and $G_{zz} = 1/z^2$ leads to Fefferman-Graham coordinates, whereas $G_{\tau z} = -1/z^2$ and $G_{zz} = 0$ to ingoing Eddington-Finkelstein chart (note that now r from (4.13) is equal to $1/z$). Of course any other (regular) choice is also acceptable[6], but these two are presumably most convenient ones for the purposes of the near-boundary and near-horizon expansion respectively. Einstein's equations

$$\mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}\,G_{AB} - \frac{6}{\mathcal{L}^2} = 0 \quad (6.8)$$

in the case of gravity dual to boost-invariant flow can be obtained directly from the action (1.1) by varying it with respect to all 5 warp-factors present in (6.7) and setting eventually G's to their gauge-fixing values. Note that varying the action only with respect to a, b and c warp-factors will not produce gravitational constraint equations, which are required for self-consistency of the approach. Outlined procedure can be generalized to more involved cases, e.g. gravity dual to linearized hydrodynamics, and is very useful when dealing with higher derivative corrections. The action containing higher derivative terms has to be evaluated on the most general metric Ansatz respecting symmetries of boundary dynamics with all gauge freedom unfixed and the result needs to be varied with respect to all available warp-factors.

The action (6.5) needs to be supplemented with boundary and counter terms, as explained in the case of universal gravity action in Chapter 1. These terms are required for holographic renormalization procedure, in particular for calculating the expectation value of the energy-momentum tensor. It turns out that when all higher derivative corrections are expressed in terms of Weyl curvature tensor and equations of motion are solved perturbatively in corrections, which is precisely the case considered here, no other boundary terms are required apart from those already present in the supergravity approximation. This is due to the fact, that Weyl tensor vanishes when evaluated on vacuum AdS metric and in asymptotically AdS spacetimes of interest the fall-off of metric is such, that higher derivative terms does not lead to

[6]Different choices will cover different parts of bulk manifold.

any new divergences. The expectation value of the energy-momentum tensor in this case is obtained from standard formula (1.15), but now with $I_{gravity}$ given by expression (6.5). It may be interesting for a reader to note that in this case, the one-point function of the energy-momentum tensor seizes to be simply a z^4-term in the near-boundary expansion, as was in supergravity approximation (1.16).

As explained above, it is sufficient to solve equations of motion for the action (6.5) up to linear order in α_2 and α_3 and to quadratic in α_1, namely

$$G_{AB} = G_{AB}^{(\mathrm{SUGRA})} + \alpha_1\, G_{AB}^{(1)} + \alpha_1^2\, G_{AB}^{(1,2)} + \alpha_2\, G_{AB}^{(2)} + \alpha_3\, G_{AB}^{(3)} + \mathrm{O}\left(\alpha_1^3,\, \alpha_1\,\alpha_2,\, \alpha_2^2,\, \alpha_3^2\right). \tag{6.9}$$

At given order, α_i's from previous orders are treated as a source with dynamical part of equations coming from variation of Einstein-Hilbert term supplemented with negative cosmological constant, which leads to very complicated and long expressions at each order. For the purposes of original publication [64], Einstein's equations were solved both for the case of the boost-invariant flow at late time and linearized hydrodynamics in the presence of quadratic and cubic interactions in the bulk up to quadratic order in α_1 and linear in α_2. The details of those calculations are skipped here and reader interested in them is directed to excellent articles outlining the foundations of the effective action method in the presence of (6.4) correction in the case of linearized [116, 118, 130] and boost-invariant hydrodynamics [117, 93]. What needs to be stressed is that there are multiple cross-checks at different levels of those calculations and solutions needed for the purposes of original publications all passed them. In particular, transport properties derived in that way agreed with known results obtained by other authors. The final result for equilibrium pressure up to desired order $\mathrm{O}\left(\alpha_1^2, \alpha_2, \alpha_3\right)$ obtained partly from direct gravitational calculations and partly taken from works by other authors [116, 117, 118, 119, 120, 19, 130, 93, 36, 74, 131] is given by

$$P = \frac{\pi^4}{2}\frac{L^3}{l_P^3}\, T^4 \left(1 + 18\alpha_1 + 24\alpha_1^2 + 24\alpha_2 + 15\alpha_3\right). \tag{6.10}$$

This formula can be related to the energy density ϵ or entropy density s, using standard relations that apply for any CFT, i.e. $\epsilon = 3P$ and $\epsilon = \frac{3}{4}\; T\; s$ (in the absence of a chemical potential). The results for transport coefficients are

$$\begin{aligned}
\frac{\eta}{s} &= \frac{1}{4\pi}\left(1 - 8\;\alpha_1 + 112\;\alpha_1^2 - 384\;\alpha_2 + 120\;\alpha_3\right),\\
\tau_\Pi T &= \frac{1}{2\pi}\left(2 - \ln 2 - 11\;\alpha_1 - 125\;\alpha_1^2 - 104\;\alpha_2 + \frac{375}{2}\;\alpha_3\right),\\
\frac{\lambda_1 T}{\eta} &= \frac{1}{2\pi}\left(1 - 2\;\alpha_1 - 146\;\alpha_1^2 - 32\;\alpha_2 + 215\;\alpha_3\right).
\end{aligned} \tag{6.11}$$

These expressions should be parametrized in terms of of dual hCFT variables, which is a subject of the following Section.

6.4 hCFT interpretation of coefficients in higher curvature expansion

Two parameters which characterize *any* $(3+1)$-dimensional CFT are its central charges, a and c, defined by conformal anomaly mentioned in Chapter 2

$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4, \tag{6.12}$$

where E_4 is the $(3+1)$-dimensional Euler density and I_4 is the square of Weyl curvature in $3+1$ dimensions. Those are given explicitly by

$$E_4 = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \qquad I_4 = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2. \tag{6.13}$$

Central charges can be computed from holographic Weyl anomaly [43, 132, 133, 134] and the result for gravity action (6.5) reads

$$a = \pi^2 \frac{\mathcal{L}^3}{l_P^3}, \qquad c = \pi^2 \frac{\mathcal{L}^3}{l_P^3} \left(1 + 8\alpha_1\right). \tag{6.14}$$

It is worth noting that in general, coefficients in holographic Weyl anomaly are given in infinite series in $l_P^2/\mathcal{L}^2$, but for higher derivative corrections parametrized by 5-dimensional Weyl curvature the result (6.14) is *exact*, i.e. it does not receive any additional contributions at higher orders in the expansion. Given the result (6.14), it is convenient to replace a and c by

$$\frac{\mathcal{L}^3}{l_P^3} = \frac{a}{\pi^2}, \qquad \alpha_1 = \frac{1}{8} \frac{c-a}{a} \equiv \frac{\delta}{8}, \tag{6.15}$$

which signals that curvature squared correction to universal gravity action has an interpretation of a correction due to *unequal* central charges of dual hCFT.

As the corresponding interaction is cubic in the Weyl tensor, α_2 naturally plays a role in defining the three-point function of the energy-momentum tensor in the dual hCFT. In general, this three-point function depends on three independent constants [135, 136]. In fact, the central charges a and c each corresponds to a certain linear combination of these parameters. Recently, it was also shown in [137] that the constants appearing in the three-point function define two new parameters with a clear physical significance in the CFT. The authors of [137] considered an "experiment" in which the energy flux was measured at null infinity after a local disturbance was created by the insertion of the energy-momentum tensor $\mathcal{O} = T^{ij}\epsilon_{ij}$ with ϵ_{ij} being a polarization tensor. The energy flux escaping at null infinity in the direction indicated by the unit vector $\vec{n}$ is then [137]

$$\langle \mathcal{E}\left(\vec{n}\right) \rangle_{\mathcal{O}} = \frac{E}{4\pi} \left\{ 1 + t_2 \left(\frac{\epsilon^*_{ij}\epsilon_{il} n_j n_l}{\epsilon^*_{ij}\epsilon_{ij}} - \frac{1}{3} \right) + t_4 \left(\frac{|\epsilon_{ij} n_i n_j|^2}{\epsilon^*_{ij}\epsilon_{ij}} - \frac{2}{15} \right) \right\}, \tag{6.16}$$

where E is the total energy of the state. The two constants appearing on the RHS of (6.16) denoted t_2 and t_4 can be used to characterize the underlying hCFT. However, recall that the three-point function contains only three independent parameters which go into defining the four constants: central charges a and c defined by (6.12), as well as t_2 and t_4 from (6.16). Thus those 4 parameters are not all independent, but rather turn out to satisfy the relation [127]

$$\frac{a}{c} = 1 - \frac{1}{6} t_2 + \frac{4}{45} t_4. \tag{6.17}$$

In the rest of the text a, c and t_4 are chosen as the independent parameters characterizing hCFT with the latter taken as most naturally connected to the cubic curvature interactions in dual gravity action. It turns out (see the original publication [64] for details) that in the parametrization (6.5)

$$t_4 = 4320\alpha_2 + \mathrm{O}\left(\alpha_1, \alpha_2, \alpha_3^2\right). \tag{6.18}$$

It is important to stress that t_4 vanishes when the dual hCFT is supersymmetric [137].

Turning to α_3, one would have to find an analogous parameter that characterizes the hCFT through the four-point function of the energy-momentum tensor. Unfortunately, the four-point function is much more difficult to analyze as it is less rigidly constrained by the symmetries of the theory than the two- or three-point functions and it depends on details of the spectrum of operators in the CFT and their couplings to the energy-momentum tensor. As a result, the four-point function is less studied and no universal hCFT parameter is known to replace the gravitational coupling α_3. However, in many string theory constructions $\alpha_3 \sim 1/\lambda^{3/2}$, where λ is the 't Hooft coupling in the dual *superconformal* gauge theory [36]. As anticipated before, at order "$\mathcal{C}^4$" in the effective gravitational action, one could write down 5 independent contractions of the Weyl tensor and hence in complete generality there would be 5 independent gravitational couplings appearing at this order [128]. In this analysis, one particular combination of interactions has been chosen, which arises naturally in constructions of type IIB superstring theory [115].

The physical parameters characterizing hCFTs with dual gravity action expanded up to quartic terms in curvature are $\{a, \delta = (c-a)/a, t_4, \alpha_3\}$ and using (6.15) and (6.18) one can re-express $\left\{P, \frac{\eta}{s}, \tau_\Pi, \lambda_1\right\}$ as

$$\begin{aligned}
P &= \frac{\pi^2}{2} a\, T^4 \left\{1 + \frac{9}{4}\delta + \frac{3}{8}\delta^2 + \frac{1}{180}t_4 + 15\alpha_3 + \mathrm{O}\left(\delta^3, t_4\delta, t_4^2, \alpha_3^2\right)\right\},\\
\frac{\eta}{s} &= \frac{1}{4\pi}\left\{1 - \delta + \frac{7}{4}\,\delta^2 - \frac{4}{45}\,t_4 + 120\,\alpha_3 + \mathrm{O}\left(\delta^3, t_4\delta, t_4^2, \alpha_3^2\right)\right\},\\
\tau_\Pi T &= \frac{1}{2\pi}\left\{2 - \ln 2 - \frac{11}{8}\,\delta - \frac{125}{64}\,\delta^2 - \frac{13}{540}\,t_4 + \frac{375}{2}\,\alpha_3 + \mathrm{O}\left(\delta^3, t_4\delta, t_4^2, \alpha_3^2\right)\right\},\\
\frac{\lambda_1 T}{\eta} &= \frac{1}{2\pi}\left\{1 - \frac{1}{4}\,\delta - \frac{73}{32}\,\delta^2 - \frac{1}{135}\,t_4 + 215\,\alpha_3 + \mathrm{O}\left(\delta^3, t_4\delta, t_4^2, \alpha_3^2\right)\right\}.
\end{aligned} \tag{6.19}$$

It needs to be noted that these expressions will receive further higher order corrections. Moreover perturbative framework requires that $a \gg 1$, $\delta \ll 1$ and $t_4 \ll 1$, as well as $\alpha_3 \ll 1$.

6.5 Consistency of the perturbative approach to higher derivative terms

Before the result (6.19) is used for any quantitative comparisons with RHIC data [36, 19], one should examine different scenarios, which might naturally arise in the dual gravitational picture. First, as discussed in [19], the gravitational couplings are typically suppressed by the ratio of the Planck scale to the AdS curvature scale with $\alpha_i \sim (l_P/\mathcal{L})^{2i}$. In this case, beyond having each $\alpha_i \ll 1$, there would be anticipated hierarchy between the couplings (6.6). On the other hand, the gravity action (6.5) overlooks the effects of dual operators other than the energy-momentum tensor. In particular, the authors [19] carefully examined that such approach is consistent up to first order in the expansion in $(l_P/\mathcal{L})^2$. However, additional considerations are required to go to higher orders when the 5-dimensional effective action includes light scalar fields coupled linearly to higher curvature terms or exactly massless scalars, as mentioned in previous Sections and explained in the following.

As an example of the first problem, one may consider a coupling of the form $\phi\,\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}$ with some massive scalar field ϕ[7]. In this case, the leading order (supergravity) solution (e.g. black brane metric) implicitly includes $\phi = 0$. However, the scalar will acquire a nontrivial profile at higher orders when the effects of the higher curvature terms are included. Note that $\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}$ term is nonzero upon evaluating on any nontrivial (other than vacuum AdS) metric and thus sources the scalar field out of its supergravity (0) value. That is, at higher orders, the dual operator acquires an expectation value in the CFT plasma. On the other hand, in the case of nonlinear coupling between the scalar and higher curvature term taking the form $\phi^\alpha\,\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}$ with $\alpha > 1$ scalar field profiled trivially would be still a solution and the problem considered above would be absent.

As the hydrodynamic properties of the plasma refer the physics at very long wavelengths, one might attempt to proceed in the case of scalar coupled linearly to higher curvature terms by integrating out this massive field. To be explicit, imagine one has the scalar action of the form

$$I_{\text{scalar}} = \frac{1}{2l_P^3}\int \mathrm{d}^5x\left\{-\partial_A\phi\,\partial^A\phi + M^2\phi^2 - 2\mathcal{L}^2\beta\,\phi\,\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}\right\}, \tag{6.20}$$

where $\beta \sim (l_P/\mathcal{L})^2$ following the discussion in [19]. If the scalar is integrated out, there is an additional contribution to the action of the form

$$\begin{aligned} I_{\text{integrated out}} = \; & \frac{1}{2l_P^3}\int \mathrm{d}^5x\Bigg\{\mathcal{L}^6\frac{\beta^2}{M^2\mathcal{L}^2}\left(\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}\right)^2 + \\ & +\mathcal{L}^8\frac{\beta^2}{M^4\mathcal{L}^4}\left(\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}\right)\nabla^2\left(\mathcal{C}_{ABCD}\mathcal{C}^{ABCD}\right)\Bigg\}. \end{aligned} \tag{6.21}$$

Now, when the scalar has a Planck scale mass, i.e. $M \sim 1/l_P$, the couplings of these higher curvature terms are suppressed in accord with the expected hierarchy. If instead one considers a scalar of a mass $M \approx 1/L$, as might arise in the Kaluza-Klein reduction of a 10-dimensional string background, the coupling constant of the new term quartic in curvature is suppressed by only $\beta^2 \sim (l_P/L)^4$ and so this term will correct the thermodynamic and the transport properties of the holographic plasma at the same order as α_1^2 and α_2. In fact, all of the higher order terms in (6.21) have the same suppression and so can be expected to contribute at the same order. Essentially this demonstrates that integrating out such scalar is not the right approach to incorporating the effects of the dual operator. Thus if one wants to work with a purely gravitational action beyond order $(l_P/L)^2$, one must impose the absence of such linear couplings in the bulk.

[7] Note that the field ϕ should be a zero mode of KK reduction to 5 dimensions (or in other words it should not be charged under the symmetries of compact manifold), since when integrating over the volume of compact manifold contribution depending on the coordinates on this manifold will average to zero. The author would like to thank Jan de Boer for pointing this out.

A subtlety appears in the case of exactly massless scalar field. In such situation this field may acquire arbitrary constant profile in the supergravity solution, which in particular may be very large. Moreover, coupling constants in the gravity theory may have some complicated dependence on this scalar field. Since the scalar field is very large in the leading order, these couplings might not be suppressed as initially assumed. Furthermore, the presence of exactly massless field(s) in the bulk seems to be interrelated with supersymmetry of the dual hCFT [64]. In turn, supersymmetry would lead to $\alpha_2 = 0$ [137], so that the interactions quartic in the curvatures would provide the next to leading order set of corrections in the perturbative expansion.

There are thus two cases, where the results (6.19) can be consistently applied in a quantitative comparison with the sQGP. The first one is the hCFT with dual effective 5-dimensional gravity action not having any light scalars coupled linearly to higher derivative terms, nor exactly massless scalars. In this case, one can truncate the string theory side of the correspondence to include only gravity. Furthermore, the expected hierarchy should hold between the gravitational couplings, so that $\alpha_2 \sim \alpha_1^2 \gg \alpha_3$ and it is then consistent to work with (6.10) and (6.11) at order $\mathrm{O}\left(\alpha_1^2, \alpha_2\right)$, while dropping the $\mathrm{O}(\alpha_3)$ contributions. The second situation is the hCFT with dual action having at least one exactly massless scalar field, which seems to imply supersymmetry [64]. As mentioned above, α_2 vanishes then and one cannot necessarily assume $\alpha_3 \ll \alpha_1$ in this scenario. Such case can be consistently described by working with the results at order $\mathrm{O}\left(\alpha_1, \alpha_3\right)$, while dropping the α_2 contributions. In both of those scenarios data may be consistently fit into the holographic framework.

6.6 Further directions

Higher derivative terms are tractable only in the Einstein gravity corner, which strongly restricts the possibilities of quantitative comparisons of hCFT thermo- and hydrodynamic properties with lattice and RHIC data. The most interesting and at the same time very difficult direction is to go beyond this limitation. In particular, there are toy models of bulk higher derivative interactions, such that perturbations on top of black brane backgrounds are quadratic in spacetime derivatives for *any* values of higher derivative couplings. The studies of thermo- and hydrodynamic properties of *toy model* hCFTs dual to those gravity actions include [120, 121, 122, 123, 124, 125, 126, 127]. Note however, that the current level of understanding of AdS/CFT does not allow to calculate interesting gauge theory quantities (e.g. η/s) for finite N_c or in the regime of intermediate couplings even in the simplest setting of UV-complete holographic gauge theory, i.e. $\mathcal{N} = 4$ SYM and to do this, further progress in the string theory is needed.

Another interesting direction is to study the holographic hydrodynamics in non-conformal settings. At first order in derivatives the bulk viscosity appears in such setups, which is however rather well-studied holographically [111, 112, 114]. At second order in gradients a bunch of new transport coefficients is produced [13] and to the author's knowledge the only article studying their properties at strong coupling is [113]. In particular, it might be interesting, but technically demanding, to formulate the fluid/gravity duality in the non-conformal case(s) of gauge/gravity duality. It may be also illuminating, but rather difficult, to understand the effects of higher derivative interactions on non-conformal gravity backgrounds dual to hydrodynamics.

Chapter 7

Boost-invariant early time dynamics from gravity

7.1 Non-Abelian plasmas in far-from-equilibrium regime

Experimental and phenomenological motivation for understanding far-from-equilibrium dynamics of non-Abelian gauge theories at strong coupling is a short thermalization time of quark-gluon plasma at RHIC, the most straightforward explanation of which attributes it to non-perturbative effects[1]. On the other hand the problem of thermalization of strongly coupled non-Abelian plasmas is intractable using any existing theoretical method apart from gauge/gravity duality and it is thus very interesting to understand it in the holographic context.

The subject of this Chapter is early time dynamics of boost-invariant flow, which is a tractable model of far-from-equilibrium dynamics in a setting of holographic conformal field theory. Original results were published in [39] as one of the very first far-from-equilibrium applications of gauge/gravity duality methods (see [21, 67] for other works on this subject). As was the case with the late time solution of Janik and Peschanski [40], Einstein's equations are going to be solved exactly in (Fefferman-Graham) radial direction in AdS and approximately in proper time, but now for $\tau = 0$[2].

The problem of early time dynamics in the same setup was addressed previously by Kovchegov and Taliotis in [141]. Those authors *assumed* certain behavior of energy density at early time and applied scaling variable method of Janik and Peschanski [40] in this domain, which eventually led them to a conclusion that early time energy density must start as a constant. On a more physical ground, it is to be expected that far-from-equilibrium part of plasma dynamics depends rather strongly on initial conditions, whereas scaling variable trick relies on unique asymptotic form of behavior. As shown in the following Section, Kovchegov and Taliotis reasoning is invalidated by a subtlety and no scaling solution exists at early time. The correct picture appearing is that $\tau = 0$ profiles of warp-factors set initial conditions both for

[1] Another possible explanation is due to collective effects in non-Abelian *perturbative* plasma [138, 139] (see [140] for a review). It may be difficult to distinguish the two scenarios, since remnants of far-from-equilibrium part of evolution are "washed-out" during hydrodynamic phase. It would be very interesting to gather theoretical predictions based on weak-coupling analysis for QCD and strong coupling analysis for holographic gauge theories and compare with relevant observables, when such are identified. The author would like to thank Stanisław Mrówczyński for an interesting discussion on this topic.

[2] Although the equations can be in principle solved in the same manner for any finite τ, a major simplification turns out to occur for $\tau = 0$.

bulk and boundary evolutions. Those profiles are however subject to gravitational constraint equations, as well as a non-singularity condition leading to highly nontrivial restrictions on possible evolutions of boundary theory's energy density. The rest of the Chapter follows along the lines of [39].

7.2 Why scaling variable does not exist at early time?

The starting point of Kovchegov and Taliotis analysis in [141] was the near-boundary expansion of warp-factors in Fefferman-Graham coordinates (4.2) with *postulated* asymptotic form of energy density[3] at $\tau = 0$ being[4]

$$\bar{\epsilon}(\tau) = \frac{1}{\tau^s}\,. \tag{7.1}$$

The authors of [141] focused on an energy density of such form because of simplicity; a priori different asymptotics (e.g. logarithmic) were also allowed at that point[5]. The value of s in (7.1) was assumed to be constrained by positivity of energy density in any reference frame, as in [40], leading to $0 \le s < 4$. Assuming finiteness of energy density per unit rapidity at $\tau = 0$ limited s further to $0 \le s \le 1$ [141][6]. In analogy with [40], if at each order of near-boundary power series (4.2) terms that dominate at $\tau = 0$ are chosen, the following resummed expression for the metric coefficient $a\,(\tau, z)$ at $\tau = 0$ is obtained

$$\frac{z^4}{\tau^s} \cdot f\left(w \equiv \frac{z}{\tau}\right) \tag{7.2}$$

with analogous ones for $b\,(\tau, z)$ and $c\,(\tau, z)$. Solving Einstein's equations in the early time scaling limit ($\tau \to 0$ keeping w fixed) leads to a unique set of solutions for each s (i.e. function $f\,(w)$ as for warp-factor a etc.), however with a complex branch cut singularity for $s > 0$. Thus the only plausible value is $s = 0$ right at the margin of the allowed range [141], i.e.

$$\epsilon(\tau) \sim const \quad \text{for} \quad \tau \to 0. \tag{7.3}$$

However, the situation when $s = 0$ is special – going back to the derivation of the early time scaling variable $w = z/\tau$, it turns out that terms in the near-boundary expansion (4.2) which lead to it are all multiplied by a factor of s. Hence for $s = 0$ all of those terms vanish and a completely *different hierarchy* of terms appears invalidating the conclusions of [141].

Indeed, if one performs for instance the power series expansion of $a(\tau, z)$ starting from (7.1) then the answer for the first three orders is

$$\begin{aligned} &-z^4\,\tau^{-s} + z^6\,\left\{\frac{1}{6}\,\tau^{-s-2}s - \frac{1}{12}\,\tau^{-s-2}s^2\right\} + \\ &+z^8\,\left\{-\frac{1}{16}\,\tau^{-2\,s}s^2 - \frac{1}{6}\,\tau^{-2\,s} + 1/6\,\tau^{-2\,s}s + \frac{1}{96}\,\tau^{-s-4}s^2 - \frac{1}{384}\,\tau^{-s-4}s^4\right\} + \ldots \end{aligned} \tag{7.4}$$

[3] In the rest of the Chapter rescaled energy density $\bar{\epsilon}$ is used instead of genuine one ϵ (4.3), but for a simplification it is still called energy density.

[4] Overall normalization of energy density plays no role in those considerations and is taken to be 1.

[5] However, the conclusions of this Chapter will make it clear that early time dynamics of boost-invariant flow is not governed by *any* scaling solution.

[6] Energy density per unit rapidity is a product of standard energy density $\epsilon(\tau)$ and volume element, which as anticipated in Chapter 3 grows linearly with proper time.

Consider now the term proportional to z^8 in (7.4). There are two structures appearing, $z^8/\tau^{(s+4)}$ and z^8/τ^{2s}. The first of these leads to the early time scaling variable proposed in [141] since it dominates for nonzero s. However its coefficient is proportional to s so for $s=0$ it is absent and the only contribution comes from the second term which is not contained in the scaling-variable analysis. The same applies obviously already to the term in $z^6/\tau^{(s+2)}$, as well as to all subsequent orders. On a related note, the energy density (7.1) with $s>0$ makes warp-factors divergent at $\tau=0$ already very close to the boundary, what from the perspective of cosmic censorship is clearly unacceptable.

Analyzing the power series solutions in more detail, one finds that for a generic early time expansion of the energy density[7]

$$\bar{\epsilon}(\tau) = \sum_{n=0}^{\infty} \bar{\epsilon}_{2n}\tau^{2n}, \tag{7.5}$$

the coefficients a_n of the near-boundary expansion of warp-factors at $\tau = 0$

$$a(\tau=0,z) = \sum_{n=0}^{\infty} a_n z^{4+2n} \tag{7.6}$$

depend on all terms $\bar{\epsilon}_{2n}$ in (7.5). This means that each of the possible initial conditions (7.6) leads to a distinct proper time evolution (7.5), where the coefficients of the two power series are linked through the Einstein's equations. The following mapping is therefore obtained

$$a(\tau=0,z) = \sum_{n=0}^{\infty} a_n z^{4+2n} \qquad \Longrightarrow \qquad \bar{\epsilon}(\tau) = \sum_{n=0}^{\infty} \bar{\epsilon}_{2n}\tau^{2n}. \tag{7.7}$$

Obviously certain nontrivial constraints limit possible gravity solutions. Among these, the non-singularity condition of the geometry plays the crucial role, as will be shown in the next Section.

7.3 Geometrical constraints on warp-factors at $\tau = 0$

Einstein's equations in the setup of gravity dual of decoupled dynamics of one-point function of energy-momentum tensor can be written in an equivalent simplified form as

$$R_{AB} + 4\,G_{AB} = 0 \tag{7.8}$$

or explicitly in Fefferman-Graham coordinates describing gravity dual to boost-invariant flow

$$\begin{aligned}
(\tau\tau) \quad &: \ddot{b}+2\ddot{c}-\frac{\dot{a}}{2}(\dot{b}+2\dot{c})+\frac{1}{2}(\dot{b}^2+2\dot{c}^2)-\frac{1}{\tau}(\dot{a}-2\dot{b}) = e^a\left\{a''-\frac{3a'}{z}+\left(\frac{a'}{2}-\frac{1}{z}\right)(a'+b'+2c')\right\},\\
(yy) \quad &: \ddot{b}-\dot{a}\dot{b}+\frac{1}{\tau}(\dot{b}-2\dot{a})+\frac{1}{2}(\dot{a}+\dot{b}+2\dot{c})\left(\dot{b}+\frac{2}{\tau}\right) = e^a\left\{b''-\frac{3b'}{z}+\left(\frac{b'}{2}-\frac{1}{z}\right)(a'+b'+2c')\right\},\\
(\perp\perp) \quad &: \ddot{c}-\dot{a}\dot{c}+\frac{\dot{c}}{2}\left(\dot{a}+\dot{b}+2\dot{c}+\frac{2}{\tau}\right) = e^a\left\{c''-\frac{3c'}{z}+\left(\frac{c'}{2}-\frac{1}{z}\right)(a'+b'+2c')\right\},\\
(z\tau) \quad &: 2\dot{b}'+4\dot{c}'+b'\left(\dot{b}+\frac{2}{\tau}\right)+2\dot{c}c'-a'\left(\dot{b}+2\dot{c}+\frac{2}{\tau}\right) = 0,\\
(zz) \quad &: a''+b''+2c''-\frac{1}{z}(a'+b'+2c')+\frac{1}{2}(a'^2+b'^2+2c'^2) = 0.
\end{aligned} \tag{7.9}$$

[7] The restriction to even powers of τ is discussed in Section 7.4.

In the above expressions the first parenthesis stands for the corresponding component of (7.8), dot for the proper time τ-derivative and prime for the z-derivative.

It is important now to analyze what are the general properties of initial conditions for the equations (7.9). Since there are five equations (7.9) for only three unknown functions, there is some redundancy in Einstein's equations, *e.g.* it is consistent to replace first three equations $(\tau\tau)$, (yy) and $(\perp\perp)$ by the combination $-(\tau\tau)+(yy)+2(\perp\perp)-e^{a}(zz)$, which reads

$$\dot{c}\left(\dot{c}+2\dot{b}+\frac{4}{\tau}\right)=e^{a}\left\{2b''+4c''+b'^{2}+3c'^{2}+2b'c'-\frac{6}{z}b'-\frac{12}{z}c'\right\}. \tag{7.10}$$

With such an replacement, the system made of three chosen equations (so (7.10) and equations $(z\tau)$ and (zz) from (7.9)) is only of the first order in τ and the required initial conditions for (proper) time evolution due to (zz) constraint equation can be given by two out of the three warp-factors evaluated at initial proper time. Moreover $(z\tau)$ component of (7.9) taking the form

$$\partial_z a-\partial_z b=\tau\cdot(\ldots), \tag{7.11}$$

indicates that at $\tau=0$, $a(\tau,z)$ and $b(\tau,z)$ can differ only by a constant. Since both functions have to vanish at $z=0$, this constant vanishes as well. Therefore

$$a(0,z)=b(0,z) \tag{7.12}$$

and gravitational initial conditions at $\tau=0$ are parametrized by a single function, which is the main reason for performing the rest of analysis exactly at $\tau=0$. It is worth stressing, that the condition (7.12) could be also obtained by demanding the absence of conical-like singularity on the light-cone $\tau=0$ for generic z.

It turns out, that the nonlinear character of the (zz) equation plays a crucial role in finding initial conditions for boost-invariant evolution. Naively, one would expect to be able to consider a small boost-invariant perturbation on top of an empty AdS_5, i.e. linearized approximation with neglecting quadratic terms in this constraint. However it turns out, as will be shown below, that even if given an infinitesimal a at $\tau=0$, the nonlinear equation for c obtained from (zz)[8] will always generate a singularity for some large but finite z! Thus linearized fluctuations cannot be used as initial conditions in the boost-invariant setup at $\tau=0$. This conclusion is even stronger, namely a singularity in metric warp-factors is going to be present at arbitrary proper time τ.

To simplify the (zz) equation it is useful to introduce the following notation

$$\begin{aligned} k_\tau(z^2) &= \frac{1}{4z}\partial_z a\,(\tau,z)\,,\\ \varsigma_\tau(z^2) &= \frac{1}{4z}\partial_z b\,(\tau,z)\,,\\ m_\tau(z^2) &= \frac{1}{4z}\partial_z c\,(\tau,z) \end{aligned} \tag{7.13}$$

with $\varsigma_0\left(z^2\right)=k_0\left(z^2\right)$ due to (7.12), but for generic $\tau>0$ different. If the constraint equation has a regular solution, radial derivatives of warp-factors are bounded in the bulk and $\upsilon_\tau,\varsigma_\tau$

[8] Note that $b\,(0,z)=c\,(0,z)$ and thus equation (zz) at $\tau=0$ indeed allows to solve for a given c or the other way around.

and w_τ vanish when $z \to \infty$. The constraint equation simplifies when written in terms of v, ς and w

$$k'_\tau + \varsigma'_\tau + 2m'_\tau + k^2_\tau + \varsigma^2_\tau + 2m^2_\tau = 0 \tag{7.14}$$

with prime now denoting z^2-derivative. Integrating (7.14) in z^2 over the whole range (i.e. from $z = 0$ to ∞) gives

$$\int_0^\infty (k'_\tau + \varsigma'_\tau + 2m'_\tau)\, \mathrm{d}(z^2) + \int_0^\infty (k^2_\tau + \varsigma^2_\tau + 2m^2_\tau)\, \mathrm{d}(z^2) = 0. \tag{7.15}$$

Note however, that first integral vanishes due to the imposed boundary conditions (i.e. assumed regularity of warp-factors), which leads to

$$\int_0^\infty (k^2_\tau + \varsigma^2_\tau + 2m^2_\tau)\, \mathrm{d}(z^2) = 0. \tag{7.16}$$

On the other hand, this equality is satisfied only for $k_\tau = \varsigma_\tau = m_\tau = 0$. Thus the only regular solution is trivial – the vacuum AdS_5. Therefore, *at any time* the metrics of interest must have a singularity at some value of z. In particular, this will be the case even at $\tau = 0$, inquiring that any nontrivial initial condition consistent with Einstein's equations will lead to a metric singularity at some value of z. The non-singularity constraint on the geometry requires that all the singularities apart from the one sitting at $z = \infty$ will be only of coordinate nature. This provides a very strong selection mechanism for the allowed initial conditions.

At $\tau = 0$ the constraint equation (7.14) can be solved exactly leading to the space of solutions parametrized fully by the single function. The trick is to introduce the linear combinations

$$\begin{aligned} v_+ &= -m_0 - k_0, \\ v_- &= m_0 - k_0 \end{aligned} \tag{7.17}$$

for which the equation (7.14) becomes algebraic for v_- (both v_- and v_+ are understood as functions of z^2). After trivial algebra one obtains

$$v_- = \sqrt{2v'_+ - v^2_+}. \tag{7.18}$$

Therefore all solutions of the initial value nonlinear constraint equations are parametrized by an arbitrary function $v_+(z^2)$. The next step is to analyze what further conditions must be imposed on $v_+(z^2)$. First, since warp-factors have to vanish as z^4 one gets

$$v_+(z^2) \sim \frac{2}{3}\epsilon_0 z^6 \quad \text{for} \quad z \sim 0. \tag{7.19}$$

Moreover, it follows from the previous argument that there is a singularity at some finite $z = z_0$. The behavior at this singularity is constrained by demanding that this would be just a coordinate singularity and not a curvature blow-up. Assuming a power-like blow-up of $v_+(z^2)$ at $z = z_0$, the regularity of the Kretschmann scalar at $\tau = 0$ leads to the conclusion, that $v_+(z^2)$ has to have a first order pole

$$v_+(z^2) \sim \frac{1}{z_0^2 - z^2} \quad \text{for} \quad z \sim z_0 \tag{7.20}$$

with residue 1.

The coordinate singularity in v_+ at $z = z_0$ translates directly into the behavior of the metric coefficients around z_0. This means that the proper time metric component has a second order zero at z_0 so that the metric at $\tau = 0$ looks like

$$ds^2 = -\frac{1}{z^2}\left(1 - \frac{z}{z_0}\right)^2 \{d\tau^2 + \tau^2 dy^2\} + \ldots + \frac{1}{z^2}dz^2 \tag{7.21}$$

in the vicinity of $z = z_0$, reminiscent of the behavior of a horizon in Fefferman-Graham coordinates. Note, however, that at $\tau = 0$ the term in curly braces becomes dx^+dx^- in contrast to a 'Schwarzschild' horizon where the corresponding structure is of the form $-(1 - z/z_0)^2 dt^2 + \ldots$ Some words of caution are in order here, since Fefferman-Graham coordinates clearly break down at $z = z_0$. If this 3-surface was indeed part of or just covered by the event horizon, then profiles of warp-factors on the range

$$z \in (0, z_0) \tag{7.22}$$

at $\tau = 0$ would form a well-posed initial value problem for subsequent (numerical) evolution of part of spacetime accessible (in a sense of cosmic censorship) from the boundary. A good probe of whether the surface $z = z_0$, $\tau = 0$ lies indeed inside a black brane region is evaluating expansion scalars (introduced in Chapter 5) within the range (7.22) looking for trapped surfaces. It turns out however that none such surface is found and in order to have a well-defined initial value problem at $\tau = 0$, the Fefferman Graham coordinate frame needs to be extended beyond the point $z = z_0$[9]. This is a subject of ongoing research project [142] which will be reported elsewhere and does not invalidate the main conclusions of original paper [39] and this Chapter[10]. Thus the rest of the Chapter follows directly the logic of [39].

It is interesting for further discussion to present explicit solutions of the constraints. One of them arises from the choice

$$v_+ = \alpha(\tan \alpha z^2 - \tanh \alpha z^2). \tag{7.23}$$

The initial metric profiles may be integrated explicitly to obtain

$$\begin{aligned} a_0(z) &= 2 \log \cos \alpha z^2, \\ c_0(z) &= 2 \log \cosh \alpha z^2. \end{aligned} \tag{7.24}$$

The above solution possesses a (coordinate – not leading to any curvature singularities) singularity at

$$z_0 = \sqrt{\frac{\pi}{2\alpha}}. \tag{7.25}$$

More generally one can parametrize v_+ in the following manner

$$v_+\left(z^2\right) = \frac{2}{3}\epsilon_0 z_0^2 \cdot \frac{z^6}{z_0^2 - z^2} V\left(z^2\right), \tag{7.26}$$

where $V(0) = 1$, $V\left(z_0^2\right) = 3/\left(2\epsilon_0 z_0^8\right)$ and otherwise is a regular function of z^2 variable for $z < \infty$[11]. One has also to ascertain that v_- obtained from (7.18) does not become complex.

[9] Analogously as in the case of standard asymptotically flat Schwarzschild black hole.

[10] i.e. the absence of scaling variable at early time, τ^2 expansion of energy density near $\tau = 0$ and relation between form of warp-factors and coefficients in early time expansion of energy density. What may change is that some of the initial conditions (7.35) have a naked singularity beyond the range of Fefferman-Graham coordinates. However, the author believes that such situation is rather unlikely. Again, this needs to be checked explicitly and is a subject of ongoing work [142].

[11] The latter requirement probably should make sure that in extended coordinate frame (as discussed above) there will be no naked singularity at finite value of (new) radial coordinate.

Once the allowed initial conditions are under control one may proceed to find dual energy density at early time.

7.4 Early time expansion of the energy density

Einstein's equations (7.8) can be solved for any energy density perturbatively in z^2. Starting with some arbitrary energy density ϵ, the first three nontrivial terms in the expansion of $a(\tau, z)$ warp factor take the form

$$\begin{aligned} a(\tau, z) &= -\bar{\epsilon}(\tau) \cdot z^4 + \left\{ -\frac{\bar{\epsilon}'(\tau)}{4\tau} - \frac{\bar{\epsilon}''(\tau)}{12} \right\} \cdot z^6 + \left\{ \frac{1}{6}\bar{\epsilon}(\tau)^2 + \frac{1}{6}\tau\bar{\epsilon}'(\tau)\bar{\epsilon}(\tau) + \frac{1}{16}\tau^2\bar{\epsilon}'(\tau)^2 \right. \\ &\left. + \frac{\bar{\epsilon}'(\tau)}{128\tau^3} - \frac{\bar{\epsilon}''(\tau)}{128\tau^2} - \frac{\bar{\epsilon}^{(3)}(\tau)}{64\tau} - \frac{1}{384}\bar{\epsilon}^{(4)}(\tau) \right\} \cdot z^8 + \cdots \end{aligned} \quad (7.27)$$

This power series can be extended to an arbitrary order in z^2 by solving Einstein's equations and the only obstructions are of a purely computational nature. Generically terms in the expansion (7.27) contain inverse powers of proper time multiplying the energy density and its derivatives. Assuming the energy density can be expanded in a regular power series around $\tau = 0$, the singular inverse powers of proper time in (7.27) will be present unless all the odd terms vanish. This requirement constrains the energy density in the early time domain to be a power series in even powers of τ, namely

$$\bar{\epsilon}(\tau) = \bar{\epsilon}_0 + \bar{\epsilon}_2\tau^2 + \bar{\epsilon}_4\tau^4 + \ldots \quad (7.28)$$

On the other hand, taking τ to be zero in equation (7.27) gives the relation between the early time energy density and the profile of $a_0(z)$, which has been signaled in Section 7.2. Expanding the initial profile of the metric in the radial AdS variable near the boundary and comparing with (7.27) at $\tau = 0$

$$a(0, z) = a_0(z) = -\bar{\epsilon}_0 \cdot z^4 - \frac{2}{3}\bar{\epsilon}_2 \cdot z^6 + \left(-\frac{\bar{\epsilon}_4}{2} - \frac{\bar{\epsilon}_0^2}{6} \right) \cdot z^8 + \ldots \quad (7.29)$$

allows one to solve for all coefficients $\bar{\epsilon}_{2i}$ sitting in (7.28). This pattern continues to any order in z (and thus τ) expansion. Note also, that each parameter in the expansion of energy density around $\tau = 0$ is an independent dimensionful quantity and in order to fully specify the initial conditions infinitely many such terms are needed. This is in stark contrast to the late time behavior, where only one dimensionful constant (denoted throughout the Thesis by Λ) appears. Physically, this is in agreement with the fact that in the thermally equilibrated final stages of Bjorken expansion all differences due to initial data should have been washed out by dissipative effects and the only parameter characterizing the flow is an overall energy scale (given e.g. as the energy density at a certain fixed large proper time τ_0).

The strategy for finding the time evolution from given initial data is to use the Einstein's equations to generate the expansion of the metric warp-factors up to sufficiently high order (the bigger, the better, but for technical reasons this has been achieved up to the order z^{84} or equivalently τ^{80} in most cases) and then for a given regular (with finite curvature evaluated on initial data for $z < \infty$, see footnote 11) initial profile generate the power series for the energy density at early time. It is worth stressing, that the presentation in this Chapter shows how beautifully the AdS/CFT correspondence sets the allowed initial conditions for *quantum* *dynamics* of gauge theory using the gravitational description. Of course, since the series (7.28) has generically a finite radius of convergence, some resummation method is needed in order to extend it to larger proper time and extract physics of interest[12].

7.5 Transition to the hydrodynamic regime

7.5.1 Resummation scheme for the energy density

The subject of this section is to provide a suitable numerical approach, relating (qualitatively) the solutions of the Einstein's equations (7.9) to physical quantities, energy density and pressures, as functions of τ up to sufficiently large τ. Although early time dynamics of the field theory is dictated by the initial conditions, after certain time the system is expected on physical grounds to settle down to local equilibrium. For τ sufficiently large the plasma should exhibit the universal hydrodynamic behavior, where the only trace of the initial conditions is given by the overall scale Λ, see (3.13). In order to track the dynamics of the system with sufficient accuracy from $\tau = 0$ to $\tau \gg 1$, numerical methods are needed. In particular the early time power series for the energy density has a finite radius of convergence and a resummation is needed in order to find its behavior for large τ.

There are two interesting physics question at this point. Firstly, whether during the evolution from some (generic) initial data at $\tau = 0$ one can observe a passage to the asymptotic perfect fluid behavior

$$\epsilon\left(\tau\right) \sim \frac{1}{\tau^{4/3}} + \ldots \tag{7.30}$$

and the second, what are the fine features of transition to equilibrium. To be more general than (7.30) one may try first to determine the asymptotic exponent s in

$$\epsilon\left(\tau\right) \sim \frac{1}{\tau^{s}} + \ldots \tag{7.31}$$

and determine how close it is to $s = 4/3$ being the expected value (see Chapters 3 and 4 for details). As explained above, it is quite difficult to answer this question as the energy density in the early time regime takes the form

$$\bar{\epsilon}\left(\tau\right) = \bar{\epsilon}_0 + \bar{\epsilon}_2\tau^2 + \ldots + \bar{\epsilon}_{2N_{cut}}\tau^{2N_{cut}} + \ldots \tag{7.32}$$

where N_{cut} is a natural number denoting the cut-off up to which the evaluation of the energy density from the initial profile in the bulk has been performed. In most cases $N_{cut} = 40$ and increasing this accuracy is difficult. Moreover series (7.32) has a finite range of convergence. In order to *estimate* the asymptotic exponent s appearing in (7.31) it is convenient to express it through a logarithmic derivative

$$s = -\lim_{\tau\to\infty}\tau\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\log\bar{\epsilon}\left(\tau\right) \tag{7.33}$$

[12] Such resummations should be understood only qualitatively and in order to get reliable profile of energy density first the issue of extending the bulk coordinate chart needs to be solved and then a numerical simulation starting from such initial data needs to be written, both of which are a subject of ongoing work [142].

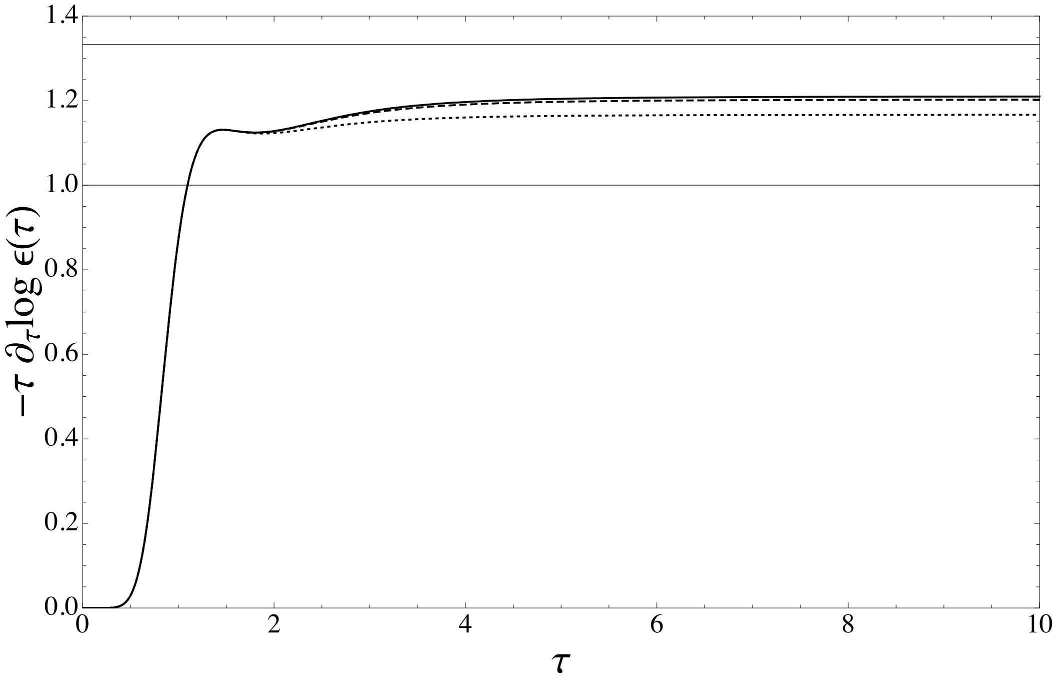


Figure 7.1: Approximate value of s obtained from the logarithmic derivative and Pade approximation for A) $N_{cut} = 32$ (dotted line), B) $N_{cut} = 40$ (dashed line) and C) $N_{cut} = 48$ (solid line) for initial profile $v_+^{(1)}(z^2)$. Two horizontal lines denote $s = 1$ (free streaming scenario) and $s = 4/3$ (perfect fluid case).

and perform a Pade approximation (of order (N_{cut}, N_{cut})) of the r.h.s. of (7.33) with $\bar{\epsilon}(\tau)$ substituted with power series (7.32) (7.32):

$$s_{\text{approx}} = \frac{s_U^{(0)} + s_U^{(2)}\tau^2 + \ldots + s_U^{(2N_{cut})}\tau^{2N_{cut}}}{s_D^{(0)} + s_D^{(2)}\tau^2 + \ldots + s_D^{(2N_{cut})}\tau^{2N_{cut}}}. \tag{7.34}$$

$s = 4/3$ corresponds then to the perfect fluid case and $s = 1$ to the so called free-streaming scenario (see [40] for more details). Of course it is not expected that (7.34) will give $s = 4/3$ exactly[13].

In the following, three examples of initial profiles satisfying the curvature non-singularity constraint are going to be considered; the first one being (7.23), the second its deformed variant, and the third one taken from the family (7.26):

$$\begin{aligned} v_+^{(1)}\left(z^2\right) &= \tan\left(z^2\right) - \tanh\left(z^2\right), \\ v_+^{(2)}\left(z^2\right) &= \tan\left(z^2\right) - \tanh\left(z^2 + \frac{z^8}{6}\right), \\ v_+^{(3)}\left(z^2\right) &= \frac{2}{3}\frac{z^6}{1-z^2}\left(1 + \frac{1}{2}z^2\right). \end{aligned} \tag{7.35}$$

The values of s for the first profile, for which energy density power series was obtained up to order τ^{100}, are $s_{\text{approx}} = 1.1667$, 1.1923, 1.2025, 1.2025 and 1.2101 respectively for $N_{cut} =$

[13] Such a Pade approximation has, by construction, a different subleading large τ behavior from viscous hydrodynamics.

32, 36, 40, 44 and 48. The approximate value of s is closer to $s = 4/3$ than to $s = 1$ for largest cut-offs (see Figure 7.1). For the second and third profile the results are inconclusive, since the corresponding energy densities are provided with worse accuracy. In the rest of the text, the asymptotic perfect fluid value $s = 4/3$ is taken as legitimate. Then, a suitable Pade resummation scheme, with the required asymptotics can be given by

$$\bar{\epsilon}^3_{\text{approx}}(\tau) = \frac{\bar{\epsilon}^{(0)}_U + \bar{\epsilon}^{(2)}_U \tau^2 + \ldots + \bar{\epsilon}^{(N_{cut}-2)}_U \tau^{N_{cut}-2}}{\bar{\epsilon}^{(0)}_D + \bar{\epsilon}^{(2)}_D \tau^2 + \ldots + \bar{\epsilon}^{(N_{cut}-2)}_D \tau^{N_{cut}+2}}. \tag{7.36}$$

where both $\bar{\epsilon}^{2i}_D$ and $\bar{\epsilon}^{2i}_U$ are obtained by expanding (7.36) around $\tau = 0$ and comparing with (7.32). Such a resummation imposes the correct asymptotic behavior, but differs, by construction, in the subleading behavior with viscous hydrodynamics.

$$\bar{\epsilon}_{\text{approx}}(\tau) = \frac{1}{\tau^{4/3}} \left\{ \# + \frac{1}{\tau^2} \cdot \# \right\}, \tag{7.37}$$

(the first subleading term scales as τ^{-2} whereas the correct scaling is $\tau^{-2/3}$). This difference is not substantial and can be cured by more refined resummation schemes[14]. Despite its simplicity, it turns out that the resummation (7.36) works pretty well (the results seem to converge well with increasing cut-off (see Figures 7.2 and 7.3) in extending the energy density beyond the convergence radius of the early time power series (which is its main task) and in providing a *qualitative* picture of dynamics.

7.5.2 Qualitative features of the approach to local equilibrium

The method (7.36) can be used to study the approximate behavior of the energy density as a function of time for large enough time to see local equilibration. Figure 7.2 shows the plots of energy density as a function of proper time for the three profiles (7.35) obtained for the highest cut-offs. The results seems to converge well for the first profile, see Figure 7.3. Energy densities obtained for these profiles differ at the initial stages, whereas in the late-time regimes both seem to approach local equilibrium[15]. A measure of the local equilibrium is the relative difference between the transverse $p_\parallel = \langle T^y_y \rangle$ and perpendicular $p_\perp = \langle T^x_x \rangle$ pressures defined as

$$\Delta p(\tau) = 1 - \frac{p_\parallel(\tau)}{p_\perp(\tau)} \tag{7.38}$$

When this quantity is close to zero, it signals the isotropization indicating local equilibrium, while a value of order one is an indication in favor of the free streaming scenario (defined by vanishing longitudinal component of the energy-momentum tensor in the boost-invariant form). Figure 7.4 shows the plot of the relative difference of pressures (7.38). It is interesting to note the rapid fall-off of the pressure difference on a scale $\tau = \mathcal{O}(1)$. This fall-off appears to be stable after different numerical checks. Interestingly enough, there is a bump which prevents the pressure to reach isotropy before $\tau = \mathcal{O}(5)$. However, Pade approximants for the pressure difference are less stable than for the energy density and the differences appear after the bump. The second profile which is a slight deformation of the first one does not seem to exhibit this bump. In any case, it would be physically interesting to check whether this phenomenon of rapid fall-off but incomplete isotropization is or not a characteristic feature

[14]Another possible issue are roots of the denominator lying within the range $(0, \infty)$. If such feature is encountered, it can be interpreted as an artifact of approximation without real physical significance.

[15]Which has been however put by hand into the resummation scheme.

of the strong coupling evolution. This issue requires using numerical methods for solving Einstein's equations and is left for future work.

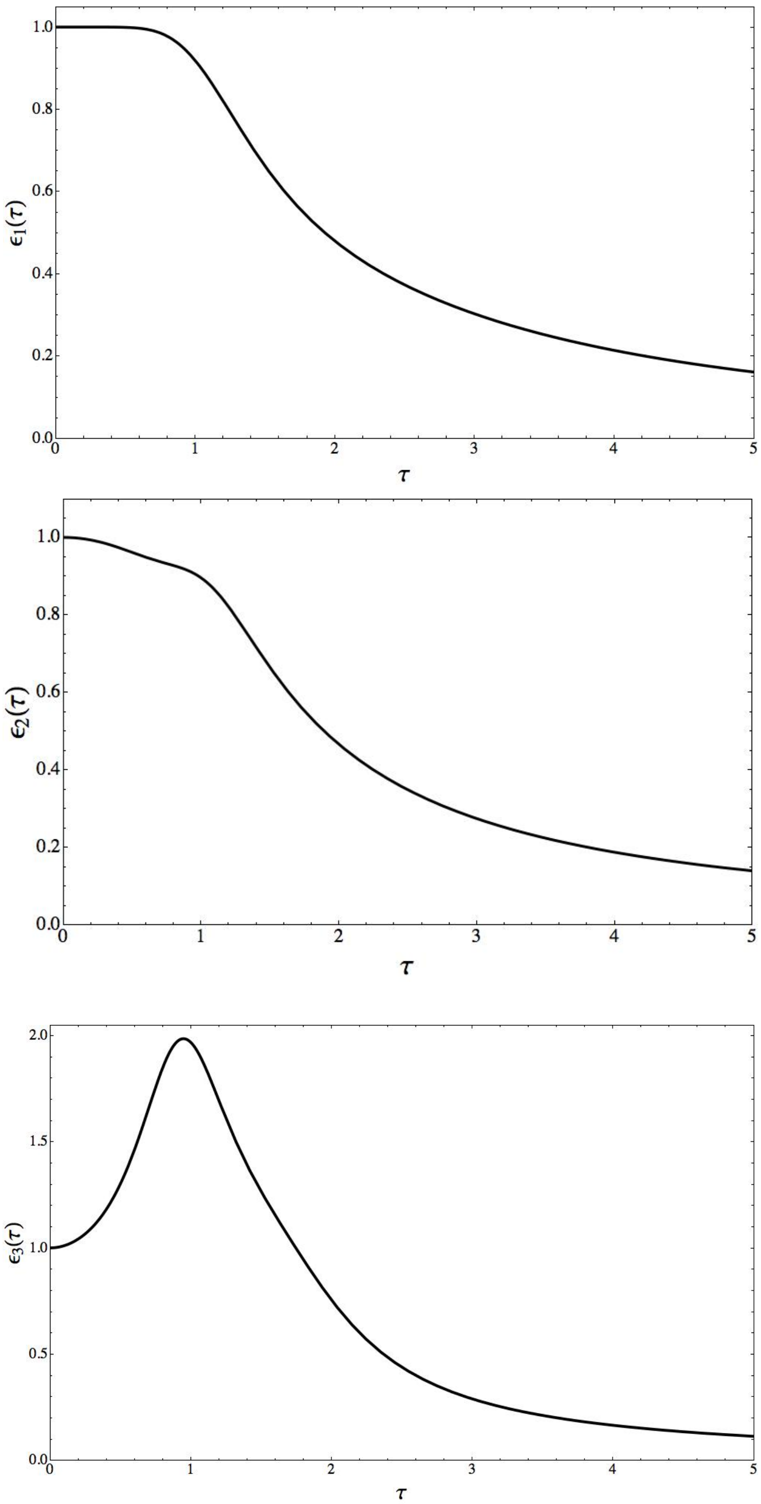
7.6 Summary

This Chapter studied the early time dynamics of boost-invariant plasma using analytical methods. The motivations for this work are both phenomenological and theoretical. On the one hand, the recent findings of the RHIC experiment suggested that locally equilibrated nuclear matter behaves as an almost perfect fluid, which presumably indicates a strongly coupled regime (at least at this stage of its evolution) of the underlying gauge field theory, *i.e.* QCD. Despite the fact that realistic dynamics of the QCD after the collision would require the understanding of rapidity dependence (*i.e.* deviations from boost-invariant dynamics, which is only approximately valid in the central rapidity region) and might be driven as well by the perturbative effects (as for the Color Glass Condensate [143] initial conditions), it is still interesting to consider the boost-invariant expansion of a conformal plasma at strong coupling as a useful toy-model for *e.g.* estimates of the thermalization time. On the other hand, theoretically, the boost-invariant plasma evolution is a workable example of a dynamical system at strong coupling, which is interesting on its own. The modern developments translates (using the AdS/CFT correspondence) the dynamics of the strongly coupled gauge theories into the evolution of higher dimensional space-time equipped with a nontrivial metric. Thus, as suggested previously by various authors (e.g. [144]) the thermalization of the excited gauge theory matter should be dual to black hole (or black brane) formation (see [21, 45] for concrete realizations of this observation), which is obtained in late time as the dual of the Bjorken hydrodynamical flow. This subject is very fresh and thus any work which may shed light on this fascinating process is valuable.

The main result of the studies reported originally in [39] and presented in this Chapter is that the boost-invariant dynamics of a strongly coupled conformal plasma is *sensitive* to the initial conditions. This contradicts the scaling hypothesis of [141], which, analogous to the late-time case [40] would indicate some uniqueness of the early time solution. In fact the scaling does not occur due to a subtlety – the *a priori* dominant scaling contributions vanish precisely in the limit $s \to 0$ (see Section 7.2 for details). The correct physical picture leads to a link between the early time expansion of energy density and the initial profile of the bulk metric. It was found as a general result that at all time, including the initial one, a singularity of at least one warp-factor should develop in the bulk providing working in Fefferman-Graham coordinates. Hence, the requirement that this does not lead to real curvature singularities already at initial time is a basic constraint on the possible initial conditions.

The analysis of the possible curvature singularities in the initial data sitting at finite Fefferman-Graham radial position fixed the early times power series for the energy density to contain only even powers of proper time. It has been shown that solving the nonlinear constraint equation in the Fefferman-Graham coordinates leads to the conclusion that the initial data must contain a coordinate singularity in the bulk of AdS, which however does not correspond to any trapped surface.

Interesting and somehow required further directions of study include extending the coordinate frame past the point $z = z_0$ and subsequent numerical investigation of the bulk evolution dual to the boost-invariant flow. This can be achieved using the methods presented in [21] and may



A)

B)

C)

Figure 7.2: A) Energy density $\epsilon_1(\tau)$ as a function of proper time τ obtained from Pade approximation for cut-off $N_{cut} = 46$ and initial profile $v_+^{(1)}(z^2)$ in the bulk; B) Energy density $\epsilon_2(\tau)$ for the second profile. C) Energy density $\epsilon_3(\tau)$ as a function of proper time τ obtained from Pade approximation for cut-off $N_{cut} = 34$ and initial profile $v_+^{(3)}(z^2)$ in the bulk.

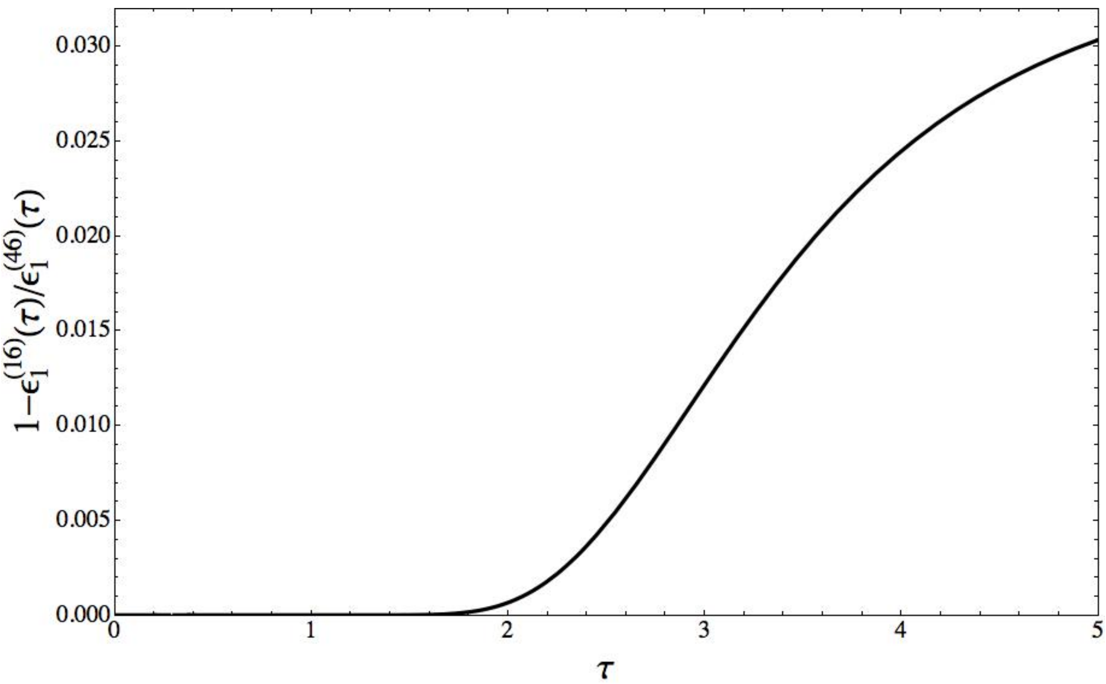


Figure 7.3: Relative difference between the energy densities for the first profile for cut-offs $N_{cut} = 16$ and $N_{cut} = 46$ does not exceed 10%.

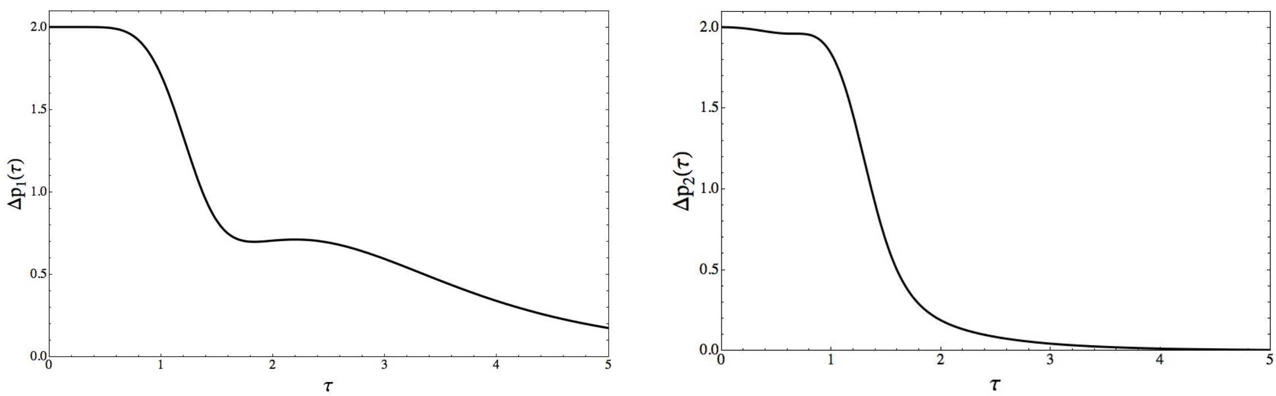


Figure 7.4: Relative difference in pressures for the first (left) and second (right) profiles – for $\tau \approx 1$ there is a rapid fall-off but (perhaps, see text) does not reach yet a complete isotropization for the first profile.

provide precise numerical results covering both early, intermediate and late time regimes (see [67] for an interesting study of boost invariant flow sourced by boundary metric perturbations). It is important to understand more qualitatively the relation between the initial conditions in the bulk and the shape of the energy density as function of proper time, which would give the more precise estimates on the thermalization time. It would be also very interesting, and in fact is a subject of ongoing work, to perform similar analysis of initial conditions in the context of ingoing Eddington-Finkelstein coordinates [145].

The methods developed in original publication and presented in this Chapter are well suited to reconsider the problem of plasma isotropization posed in [87] using analytical methods (note that anisotropic energy-momentum tensor should reach the equilibrium exponentially fast, whereas hydrodynamic evolution leaves a power-like tail). Finally the most interesting, yet highly non-trivial, extensions of the AdS/CFT program for the dynamical evolution of a plasma are the studies of the initial conditions such as shock waves collision using the dual gravity picture (see for instance [146, 147] for some preliminary attempts).

Overview

Applied gauge/gravity duality

The AdS/CFT correspondence provides a novel perspective on both quantum field theories and (quantum) gravity. The weak/strong nature of the conjectured equivalence makes it an invaluable method to understand the strongly coupled side (being either the string theory in the bulk or quantum field theory on the boundary) using the weakly coupled dual description. In particular, *applied string theory* uses Einstein gravity as a masterfield description of certain gauge theories at large number of colors and strong coupling ($\mathcal{N} = 4$ super Yang-Mills in 3+1 dimensions being the primary example) to understand non-perturbative quantum field theory dynamics in setups inspired by the real-world questions. Up to date, there were two motivations to undertake that path of research: non-perturbative physics of Quantum Chromodynamics and condensed matter phenomena possibly described by some effective strongly coupled quantum field theories. The results obtained so far do not allow for quantitative explanations of the experiments. Rather than that, they should be and in fact are understood as order of magnitude estimates[16] or as clues providing insights or predicting new phenomena in strongly coupled systems existing in nature[17].

The topic of the Thesis was to use existing methods within the gauge/gravity duality and develop new ones to understand QCD-inspired time-dependent phenomena in strongly coupled holographic quantum field theories. Among all gauge/gravity dualities the simplest ones are those, which hold for conformal field theories and such were the objects of this study. It is clear by now, that any holographic conformal field theory at large number of colors and strong coupling has a universal sector of its dynamics fully specified in terms of one-point function of the energy-momentum tensor [45]. The gravity dual picture of this dynamics is described by the universal gravity action consisting of Einstein-Hilbert term supplemented with the negative cosmological constant. This means that there is a very rich (dual) physics hidden within the solutions of classical gravity in asymptotically anti-de Sitter spacetimes. In particular, analytic and numerical methods already shed some light on such important questions as transport properties of strongly coupled gauge theories or isotropisation and thermalization of excited strongly coupled matter [21, 67], just to name a few.

[16] e.g. the ratio of the shear viscosity to the entropy density of strongly coupled Quark Gluon Plasma is of the order of magnitude of the famous string theory result $\frac{\eta}{s} = \frac{1}{4\pi}$ in natural units.

[17] e.g. suggesting that it is possible to see the effects of QCD axial anomaly in non-central heavy ion collisions [14, 15] or going beyond the standard Landau theory of Fermi surfaces with possible implications on the physics of strange metals [148, 149, 150].

The role of cosmic censorship

The universal gravity action consisting of Einstein-Hilbert term and negative cosmological constant is a low energy limit of complicated dynamics of string theory. Consistent truncation of stringy physics is possible only if the bulk system will remain in the low energy regime at all times. The gravity is known to produce curvature singularities and it is crucial for the consistency of the approach that those singularities are covered by event horizons and are thus disconnected from the dynamics. Naked singularities in the bulk are interpreted as unphysical configurations in the dual quantum field theory description. An example of naked (gravitational) singularity in the context of the AdS/CFT correspondence is the one present in the gravity description of static anisotropic plasma [87]. The singularity ceases to be there only if the anisotropy parameter is taken to be zero, which leads to AdS-Schwarzschild black hole. This means that the only static plasma configuration in the boundary field theory in the universal sector of decoupled dynamics of the energy-momentum tensor is the isotropic plasma in the global equilibrium. This does not mean that there are not any other physical configurations from other sectors (such that both the energy-momentum tensor and some other operator(s) acquire(s) expectation value(s)) in which the plasma is anisotropic. In particular, in a system of gravity with negative cosmological constant coupled to SU(2) gauge vector field, there is a phase transition for sufficiently low temperatures to the anisotropic case with the anisotropy in the energy-momentum tensor sourced by the black hole non-Abelian vector hair [88].

Different coordinate charts in AdS and nonsingularity

The general covariance of Einstein gravity means that any coordinate chart in the bulk of AdS will do. Different coordinates cover different patches of the spacetime and might be more convenient for different purposes. The standard coordinates when dealing with near-boundary behavior of AdS spacetime are Fefferman-Graham ones (see [37] and references therein), whereas the example of ones which are explicitly regular in the near-horizon region is given by ingoing Eddington-Finkelstein coordinates (see [35] for a discussion on those). The nonsingularity condition should hold in any coordinate frame, but this Thesis demonstrates that it might be not always transparent.

Summary of results

Since Einstein's equations are a very complicated system of nonlinear partial differential equations, it is important to have an example of the gravitational dynamics, which is simple enough for efficient treatment and at the same time is capable of describing some of the physics of interest. Such an example is given by the boost-invariant flow on the quantum field theory side [33], which is a $(1+1)$-dimensional dynamics of plasma with the boost-invariance along the expansion axis. The assumption of boost-invariance makes the problem effectively one-dimensional (in a sense that quantum field theory observables depend only on single coordinate – proper-time) and simultaneously allows for hydrodynamic regime. It is possible then to construct the gravity dual solutions analytically at late and early proper-time. These constructions, as well as analysis of their properties combined into the body of this Thesis.

Chapter 4. Late time solution, its regularity and hydrodynamics: Prior attempts to construct the gravity dual of boost-invariant flow of hCFT plasmas suffered from apparent curvature singularities in the late time expansion [68]. These results lead to the conjecture that boost-invariant flow cannot be realized in hCFTs with supergravity dual [16, 17]. The Letter [18] showed how these problems could be resolved by a different choice of expansion parameter with a singular coordinate transformation in the bulk relating this and previous approach. This work also clarified the meaning of the non-singularity argument used by authors of [40] and ensured that the gravity dual to the boost-invariant flow is a reliable tool for computing certain transport coefficients beyond supergravity dual [117, 93, 64]. Moreover, it opened the possibility to study the global structure of the boost-invariant spacetime at late time [90, 49].

Chapter 5. Horizons in the gravity dual to the boost-invariant flow and hydrodynamic entropy current: Article [49] generalized the framework of slowly evolving horizons [151, 106] to the case of black branes in asymptotically anti-de Sitter spacetimes in arbitrary dimensions. The results were used to analyze the behavior of both event and apparent horizons in the gravity dual to boost-invariant flow. These considerations were motivated by the fact that at second order in the gradient expansion the hydrodynamic entropy current in the dual Yang-Mills theory appeared to contain an ambiguity [13]. This ambiguity, in the case of boost-invariant flow, was linked with a similar freedom on the gravity side leading to a phenomenological definition of the entropy of black branes. Some insights on fluid/gravity duality and the definition of entropy in a time-dependent setting were also elucidated. This work was one of the first examples of possible direct physical interpretation of quasilocal notions of horizons (see [48] for a review). It might be also relevant for numerical studies of gravitational solutions dual to far-from-equilibrium quantum field theory configurations (such as [21, 67]), because quasilocal horizons provide a natural cut-off for numerical integration in the radial direction. A follow-up project, which generalizes the results of [49] to arbitrary hydrodynamic flow within the fluid/gravity duality, is the subject of ongoing work [108].

Chapter 6. sQGP as hCFT: The paper [64] examined the proposal to make qualitative comparisons between the strongly coupled quark-gluon plasma and holographic descriptions of conformal field theory. In particular, leading corrections from higher curvature terms in the dual gravity theory to certain transport coefficients appearing in second order hydrodynamics were studied. The applicability of these results to quantitative comparisons with the sQGP was also discussed. The approach started in [19] and developed further in [64] is one of the very few attempts to treat holographic conformal field theories as an effective description of real-world phenomena.

Chapter 7. Far-from-equilibrium dynamics of boost-invariant plasma: Article [39] studied the boost-invariant dynamics of a strongly-coupled conformal plasma in the regime of early proper-time. It was one of the first attempts to understand far-from-equilibrium dynamics at strong coupling using the AdS/CFT correspondence and the first one which studied setting the initial conditions for field theory dynamics using the gravitational description. In particular, it was shown, in contrast with the late-time expansion and contrary to claims by other authors [141], that a scaling solution at early time does not exist. The boundary dynamics in this regime turned out to depend on initial conditions encoded in the bulk behavior of a Fefferman-Graham metric coefficient at initial proper-time. The relation between the early-time expansion of the energy density and initial conditions in the bulk of AdS was provided and as a general result it was proven that a singularity of some metric coefficient in Fefferman-Graham frame exists at all times. Requiring that this singularity at $\tau = 0$ is a mere coordinate singularity without the curvature blow-up provided constraints on the possible boundary dynamics. Subsequent analytic and numerical approximations revealed subsequent reach dynamics. Numerical studies of the gravity dual to boost-invariant flow starting from some regular initial conditions are subjects of ongoing projects [142, 145].

Future directions

The dynamics of boost-invariant flow at early and transient time is one the very first examples of the use of string theory methods to understand far-from-equilibrium physics of non-perturbative quantum field theories. Further far-from-equilibrium applications of gauge/gravity duality are the most interesting future line of research. Future developments will need combined analytic and numerical methods and may be important for understanding of such diverse systems as early time phases of heavy ion collision and thermalization of the resulting fireball, singularity formation in the bulk of AdS or even cosmology of the early universe (dynamics of plasma on the expanding background). One particularly interesting application of gauge/gravity duality lies in the study of collisions of gravitational shock waves (see [146, 147, 152, 153] for preliminary attempts or [154, 155, 156] for estimates of entropy production in heavy ion collisions based on the area of apparent horizon prior to collision of gravitational waves). In the dual QFT language, the collision of gravitational waves maps into the problem of collisions of lumps of matter which, superficially at least, can be made to mimic collisions of nuclei in QCD. The power of the gravitational description in this and similar setups is that it allows for tracking of observables during the complete time evolution of the system – from the initial far-from-equilibrium dynamics to the late-time onset of hydrodynamics. However, it has to be borne in mind that some of pre-equilibrium RHIC physics might be driven by perturbative phenomena.

Another interesting direction is to extend the phenomenological constructions of "effective hCFT of sQGP" including deviations from conformality in more systematic fashion[18], as well as going beyond the Einstein gravity corner (which may however need better understanding of the gauge/gravity duality itself). On equal footing, recent progress in obtaining simplest transport properties of gluodynamics from lattice might be a signal, that for some QCD quantities the gauge/gravity duality is going to be replaced by more and more capable computer simulations [157, 158] (see [159] for a review).

Very interesting, but somehow disconnected from the applications discussed in the Thesis, are attempts to understand condensed matter phenomena at strong coupling in terms of gravity dual description. Until now, almost all of the effort focused on static properties or linearized dynamics and it would be very tempting to use the machinery developed for studying QCD-inspired setups within the AdS/CFT correspondence to model real-time nonlinear dynamics of collective phenomena at strong coupling.

It is peculiar that string theory – the unique framework offering the possibility of UV completion of gravity and Standard Model interactions – is closest to the real-world applications via the holographic correspondence. In particular, the gauge/gravity duality provided a tractable arena for the controlled study of strongly coupled dynamics in non-Abelian gauge theories. Such calculations involved gravitational collapse and ensuing dynamics of black holes in the bulk of higher dimensional (in simplest examples asymptotically Anti-de Sitter) spacetimes. The importance of the results obtained was twofold. First, gauge/gravity duality is a powerful tool which can be used to systematically understand the dynamics of strongly coupled Quantum Field Theories from classical gravitational dynamics and some of its results (e.g. transport properties at strong coupling) have proved to be useful. Second, dual (gravitational) calculations which reveal the dynamics of strongly coupled gauge theories describe the evolution of black branes and may teach us something about real-world black holes or gravity in general. Specifically, they could lead to progress in string theory by addressing physical phenomena where quantum gravity is expected to be important. Such advances might bring us closer to understanding of real-world on smallest scales, which has been the ultimate task of string theory for last 30 years.

[18] e.g. understand the fluid/gravity duality up to second order in gradients in non-conformal case.

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